SIMILAR TRIANGLES AND AREA RATIOS FOR AMC 10: SOLUTIONS

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1. Solutions

Beginner.

1. (1951 AHSME, Problem #29) Of the following sets of data, the only one that does not determine the shape of a triangle is:

(A) The ratio of two sides and the included angle
(B) The ratios of the three altitudes **(C)** The ratios of the three medians
(D) The ratio of the altitude to the corresponding base **(E)** Two angles

Strategy. Two triangles have the same shape exactly when they are similar. Match each set of data with a criterion of Theorem 1.3.1 of the handout, and look for two triangles that are not similar but share the remaining data.

Solution. Data (A) fixes the shape by SAS, and data (E) by AA.

For (B), a triangle with area K and sides a, b, c has altitudes $\frac{2K}{a}$, $\frac{2K}{b}$ and $\frac{2K}{c}$, since K is half of each side times the altitude to it. So the ratios of the altitudes fix the ratios $\frac{1}{a} : \frac{1}{b} : \frac{1}{c}$, hence the ratios $a : b : c$, and SSS applies.

For (C), the median m_a to the side a satisfies $4m_a^2 = 2b^2+2c^2-a^2 = 2(a^2+b^2+c^2)-3a^2$, and similar equations hold for m_b and m_c. Adding the three equations gives $a^2+b^2+c^2 = \frac{4}{3}(m_a^2+m_b^2+m_c^2)$, and then

$$9a^2 = 4\big(2m_b^2+2m_c^2-m_a^2\big),$$

with similar formulas for b^2 and c^2. So when the medians are multiplied by a common factor, the sides are multiplied by the same factor: the ratios of the medians fix the ratios of the sides, and SSS applies again.

Data (D) does not fix the shape. An isosceles right triangle with legs 2 and an isosceles triangle with base 2 and height 2 both have altitude 2 to a side of length 2. Their sides are 2, 2, $2\sqrt{2}$ and 2, $\sqrt{5}$, $\sqrt{5}$, which are not proportional, so the triangles are not similar.

Answer: (D) The ratio of the altitude to the corresponding base. □

2. (1965 AHSME, Problem #12) A rhombus is inscribed in triangle ABC in such a way that one of its vertices is A and two of its sides lie along AB and AC. If $AC = 6$ inches, $AB = 12$ inches, and $BC = 8$ inches, the side of the rhombus, in inches, is:

(A) 2 **(B)** 3 **(C)** $3\frac{1}{2}$ **(D)** 4 **(E)** 5

Strategy. The side of the rhombus opposite its side along AC is parallel to AC, so it cuts triangle ABC parallel to AC.

Solution. Let the rhombus be $ADFE$ with D on AB, E on AC and F on BC, and let x be its side. Opposite sides of a rhombus are parallel, so $DF \parallel AE$, that is, $DF \parallel AC$. In triangle BAC, the points D and F lie on the sides BA and BC, so by Theorem 1.2.2 of the handout

$$\frac{DF}{AC} = \frac{BD}{BA}, \qquad \frac{x}{6} = \frac{12-x}{12}.$$

Then $2x = 12 - x$, so $x = 4$. The length $BC = 8$ is not needed.

Answer: (D) 4. □

3. (1972 AHSME, Problem #18) Let $ABCD$ be a trapezoid with the measure of base AB twice that of base DC, and let E be the point of intersection of the diagonals. If the measure of diagonal AC is 11, then that of segment EC is equal to:

(A) $3\frac{2}{3}$ **(B)** $3\frac{3}{4}$ **(C)** 4 **(D)** $3\frac{1}{2}$ **(E)** 3

Strategy. Theorem 2.2.1 of the handout gives the ratio in which E divides AC.

Solution. By Theorem 2.2.1, $AE : EC = AB : DC = 2 : 1$. So EC is one third of AC:

$$EC = \frac{11}{3} = 3\tfrac{2}{3}.$$

Answer: (A) $3\frac{2}{3}$. □

4. (1958 AHSME, Problem #19) The sides of a right triangle are a and b and the hypotenuse is c. A perpendicular from the vertex divides c into segments r and s, adjacent respectively to a and b. If $a : b = 1 : 3$, then the ratio of r to s is:

(A) $1 : 3$ **(B)** $1 : 9$ **(C)** $1 : 10$ **(D)** $3 : 10$ **(E)** $1 : \sqrt{10}$

Strategy. Theorem 3.1 of the handout relates each leg to the segment of the hypotenuse next to it.

Solution. The segment r is next to the leg a, and s is next to b. By Theorem 3.1, $a^2 = r \cdot c$ and $b^2 = s \cdot c$. Dividing,

$$\frac{r}{s} = \frac{a^2}{b^2} = \left(\frac{1}{3}\right)^2 = \frac{1}{9}.$$

Answer: (B) $1 : 9$. □

5. (1962 AHSME, Problem #16) Given rectangle R_1 with one side 2 inches and area 12 square inches. Rectangle R_2 with diagonal 15 inches is similar to R_1. Expressed in square inches the area of R_2 is:

(A) $\frac{9}{2}$ **(B)** 36 **(C)** $\frac{135}{2}$ **(D)** $9\sqrt{10}$ **(E)** $\frac{27\sqrt{10}}{4}$

Strategy. Find the diagonal of R_1. The ratio of the diagonals is the ratio of similarity, and the areas are in the ratio of its square.

Solution. The other side of R_1 is $\frac{12}{2} = 6$, so its diagonal is $\sqrt{2^2 + 6^2} = \sqrt{40} = 2\sqrt{10}$. By Theorem 4.2.2 of the handout, the ratio of similarity is the ratio of corresponding diagonals, $k = \frac{15}{2\sqrt{10}}$, and the area of R_2 is

$$k^2 \cdot 12 = \frac{225}{40} \cdot 12 = \frac{135}{2}.$$

Answer: (C) $\frac{135}{2}$. □

Intermediate.

6. (1971 AHSME, Problem #28) Nine lines parallel to the base of a triangle divide the other sides each into 10 equal segments and the area into 10 distinct parts. If the area of the largest of these parts is 38, then the area of the original triangle is:

(A) 180 **(B)** 190 **(C)** 200 **(D)** 210 **(E)** 240

Strategy. Each line cuts off, at the vertex opposite the base, a triangle similar to the whole one. Write the area of each part as the difference of two such triangles.

Solution. Let K be the area of the triangle. Counting from the vertex opposite the base, the j-th line cuts off a triangle similar to the whole one with ratio $\frac{j}{10}$, by Theorem 1.2.2

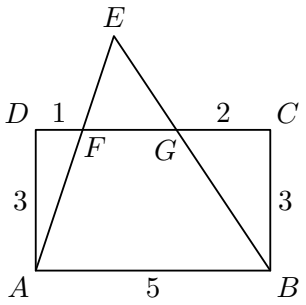
of the handout, so its area is $\frac{j^2}{100}K$ by Theorem 4.1.1. The j-th part, between lines $j-1$ and j (with the vertex as line 0 and the base as line 10), therefore has area

$$\frac{j^2-(j-1)^2}{100}K = \frac{2j-1}{100}K.$$

This grows with j, so the largest part is the one next to the base, with area $\frac{19}{100}K$. From $\frac{19}{100}K = 38$, $K = 200$.

Answer: (C) 200. □

7. (2003 AMC 10B, Problem #20) In rectangle $ABCD$, $AB = 5$ and $BC = 3$. Points F and G are on $\overline{CD}$ so that $DF = 1$ and $GC = 2$. Lines AF and BG intersect at E. Find the area of $\triangle AEB$.



(A) 10 **(B)** $\frac{21}{2}$ **(C)** 12 **(D)** $\frac{25}{2}$ **(E)** 15

Strategy. The segment FG is parallel to AB, so triangle EFG is similar to triangle EAB. Their heights from E differ by the height of the rectangle.

Solution. The segment $FG = DC - DF - GC = 5 - 1 - 2 = 2$ is parallel to AB and shorter, so the lines AF and BG, the legs of trapezoid $ABGF$, meet beyond FG, as in the figure, and F and G lie on the sides EA and EB of triangle EAB. By Theorem 1.2.2 of the handout, $\triangle EAB \sim \triangle EFG$ with ratio $\frac{FG}{AB} = \frac{2}{5}$. Let h be the height of triangle EAB from E. The lines FG and AB are 3 apart, so the height of triangle EFG from E is $h-3$. By Theorem 4.1.1,

$$\frac{h-3}{h} = \frac{2}{5}, \qquad 5h - 15 = 2h, \qquad h = 5,$$

and $[AEB] = \frac{1}{2} \cdot 5 \cdot 5 = \frac{25}{2}$.

Answer: (D) $\frac{25}{2}$. □

8. (2016 AMC 10A, Problem #19) In rectangle $ABCD$, $AB = 6$ and $BC = 3$. Point E between B and C, and point F between E and C are such that $BE = EF = FC$. Segments $\overline{AE}$ and $\overline{AF}$ intersect $\overline{BD}$ at P and Q, respectively. The ratio $BP : PQ : QD$ can be written as $r : s : t$ where the greatest common factor of r, s, and t is 1. What is $r + s + t$?

(A) 7 **(B)** 9 **(C)** 12 **(D)** 15 **(E)** 20

Strategy. The segments BE and BF are parallel to AD, and BD crosses each of AE and AF. Theorem 2.2.1 of the handout, in the trapezoids $ABED$ and $ABFD$, locates P and Q on BD.

Solution. Since $BE = EF = FC$ and $BC = 3$, $BE = 1$ and $BF = 2$; also $AD = BC = 3$. In trapezoid $ABED$ the bases BE and AD are parallel and the diagonals AE and BD

meet at P, so by Theorem 2.2.1

$$\frac{BP}{PD} = \frac{BE}{AD} = \frac{1}{3}, \qquad \text{so} \qquad BP = \tfrac{1}{4}\,BD.$$

In trapezoid $ABFD$, with bases BF and AD and diagonals meeting at Q,

$$\frac{BQ}{QD} = \frac{BF}{AD} = \frac{2}{3}, \qquad \text{so} \qquad BQ = \tfrac{2}{5}\,BD.$$

As fractions of BD,

$$BP = \frac{1}{4} = \frac{5}{20}, \qquad PQ = \frac{2}{5} - \frac{1}{4} = \frac{3}{20}, \qquad QD = 1 - \frac{2}{5} = \frac{12}{20},$$

so $BP : PQ : QD = 5 : 3 : 12$. These numbers have greatest common factor 1, and $r + s + t = 5 + 3 + 12 = 20$.

Answer: (E) 20. □

9. (2005 AMC 10B, Problem #23) In trapezoid $ABCD$ we have $\overline{AB}$ parallel to $\overline{DC}$, E as the midpoint of $\overline{BC}$, and F as the midpoint of $\overline{DA}$. The area of $ABEF$ is twice the area of $FECD$. What is $\dfrac{AB}{DC}$?

(A) 2 **(B)** 3 **(C)** 5 **(D)** 6 **(E)** 8

Strategy. The midline FE cuts the trapezoid into two trapezoids of the same height. Corollary 2.1.2 of the handout gives the length of FE.

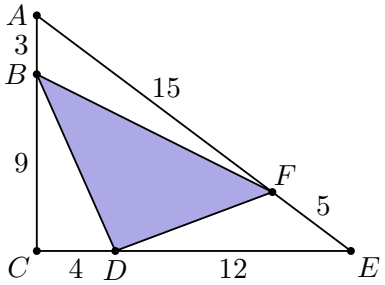
Solution. Let $AB = a$, $DC = b$, and let h be the height of $ABCD$. By Corollary 2.1.2, FE is parallel to the bases and $FE = \frac{a+b}{2}$. The line FE is halfway between the bases: if D' is the foot of the perpendicular from D to line AB, then in triangle DAD' the line FE passes through the midpoint F of DA and is parallel to AD', so by Theorem 2.1.1 it meets DD' at its midpoint. (If $D' = A$, then DA is itself perpendicular to the bases and F is its midpoint.) So $ABEF$ and $FECD$ are trapezoids with height $\frac{h}{2}$, and

$$[ABEF] = \frac{h}{4}\Big(a + \frac{a+b}{2}\Big) = \frac{(3a+b)h}{8}, \quad [FECD] = \frac{h}{4}\Big(\frac{a+b}{2} + b\Big) = \frac{(a+3b)h}{8}.$$

The condition $[ABEF] = 2[FECD]$ gives $3a + b = 2(a + 3b)$, so $a = 5b$, and $\frac{AB}{DC} = 5$.

Answer: (C) 5. □

10. (2004 AMC 10B, Problem #18) In right triangle $\triangle ACE$, we have $AC = 12$, $CE = 16$, and $EA = 20$. Points B, D, and F are located on AC, CE, and EA, respectively, so that $AB = 3$, $CD = 4$, and $EF = 5$. What is the ratio of the area of $\triangle BDF$ to that of $\triangle ACE$?



(A) $\dfrac{1}{4}$ **(B)** $\dfrac{9}{25}$ **(C)** $\dfrac{3}{8}$ **(D)** $\dfrac{11}{25}$ **(E)** $\dfrac{7}{16}$

Strategy. Triangle BDF is triangle ACE without three corner triangles, and each corner triangle shares an angle with ACE.

Solution. Let $K = [ACE]$. Then $BC = 9$, $DE = 12$ and $FA = 15$. By Theorem 5.1.1 of the handout, the corner triangles have areas

$$[ABF] = \frac{AB}{AC} \cdot \frac{AF}{AE} K = \frac{3}{12} \cdot \frac{15}{20} K = \frac{3}{16} K,$$
$$[CDB] = \frac{CD}{CE} \cdot \frac{CB}{CA} K = \frac{4}{16} \cdot \frac{9}{12} K = \frac{3}{16} K,$$
$$[EFD] = \frac{EF}{EA} \cdot \frac{ED}{EC} K = \frac{5}{20} \cdot \frac{12}{16} K = \frac{3}{16} K.$$

So $[BDF] = K - 3 \cdot \frac{3}{16} K = \frac{7}{16} K$, and the ratio is $\frac{7}{16}$. The right angle is not needed.

Answer: (E) $\dfrac{7}{16}$. □

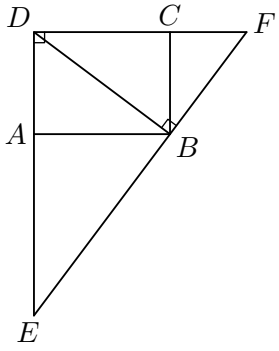
Advanced.

11. (2009 AMC 10A, Problem #17) Rectangle $ABCD$ has $AB = 4$ and $BC = 3$. Segment EF is constructed through B so that $EF \perp DB$, and A and C lie on DE and DF, respectively. What is EF?

(A) 9 **(B)** 10 **(C)** $\dfrac{125}{12}$ **(D)** $\dfrac{103}{9}$ **(E)** 12

Strategy. Triangles DBE and DBF have right angles at B, and BA and BC are their altitudes to the hypotenuses DE and DF. Use Theorem 3.1 of the handout in each.

Solution. The diagonal is $DB = \sqrt{4^2 + 3^2} = 5$. The figure shows the configuration.



Triangle DBE has a right angle at B, because $EF \perp DB$, and A lies on its hypotenuse DE with $BA \perp DE$, because $\angle BAD = 90°$. So BA is the altitude to the hypotenuse, and Theorem 3.1 gives

$$DB^2 = DA \cdot DE, \qquad \text{so} \qquad DE = \frac{25}{3}, \qquad EA = \frac{25}{3} - 3 = \frac{16}{3},$$

and $BE^2 = EA \cdot ED = \frac{16}{3} \cdot \frac{25}{3} = \frac{400}{9}$, so $BE = \frac{20}{3}$. In the same way, triangle DBF has a right angle at B and altitude BC to its hypotenuse DF, so

$$DB^2 = DC \cdot DF, \qquad \text{so} \qquad DF = \frac{25}{4}, \qquad FC = \frac{25}{4} - 4 = \frac{9}{4},$$

and $BF^2 = FC \cdot FD = \frac{9}{4} \cdot \frac{25}{4} = \frac{225}{16}$, so $BF = \frac{15}{4}$. Therefore

$$EF = EB + BF = \frac{20}{3} + \frac{15}{4} = \frac{125}{12}.$$

As a check, triangle DEF has a right angle at D and altitude DB, and $EB \cdot BF = \frac{20}{3} \cdot \frac{15}{4} = 25 = DB^2$, as Theorem 3.1 requires.

Answer: (C) $\dfrac{125}{12}$. □

12. (1966 AHSME, Problem #32) Let M be the midpoint of side AB of triangle ABC. Let P be a point on AB between A and M, and let MD be drawn parallel to PC and intersecting BC at D. If the ratio of the area of triangle BPD to that of triangle ABC is denoted by r, then:

(A) $\dfrac{1}{2} < r < 1$ depending upon the position of P

(B) $r = \dfrac{1}{2}$ independent of the position of P

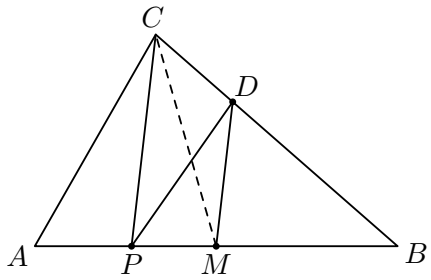
(C) $\dfrac{1}{2} \leq r < 1$ depending upon the position of P

(D) $\dfrac{1}{3} < r < \dfrac{2}{3}$ depending upon the position of P

(E) $r = \dfrac{1}{3}$ independent of the position of P

Strategy. Split triangle BPD along MD. Since $MD \parallel PC$, Theorem 1.2.1 of the handout replaces P by C in one of the two pieces.

Solution. On side AB the points lie in the order A, P, M, B, so the segment MD splits triangle BPD into triangles BMD and MPD, as in the figure.



Triangles MDP and MDC have the common base MD. The line MD crosses line AB at M, between B and P, and crosses line BC at D, between B and C, so P and C both lie on the side of line MD away from B. Since $PC \parallel MD$, Theorem 1.2.1 gives $[MDP] = [MDC]$. Hence

$$[BPD] = [BMD] + [MDP] = [BMD] + [MDC] = [BMC].$$

The segment CM is a median of triangle ABC, and a median divides a triangle into two triangles of equal area, so $[BMC] = \frac{1}{2}[ABC]$. Therefore $r = \frac{1}{2}$ for every position of P.

Answer: (B) $r = \dfrac{1}{2}$ independent of the position of P. □

13. (2009 AMC 10A, Problem #23) Convex quadrilateral $ABCD$ has $AB = 9$ and $CD = 12$. Diagonals AC and BD intersect at E, $AC = 14$, and $\triangle AED$ and $\triangle BEC$ have equal areas. What is AE?

(A) $\dfrac{9}{2}$ **(B)** $\dfrac{50}{11}$ **(C)** $\dfrac{21}{4}$ **(D)** $\dfrac{17}{3}$ **(E)** 6

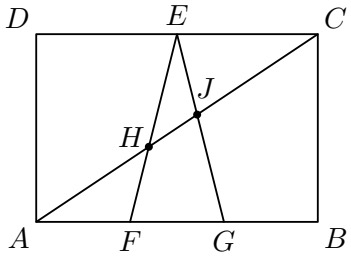
Strategy. Add the area of triangle ABE to both of the equal areas. The result shows that C and D are equally far from line AB, and then Theorem 2.2.1 of the handout applies.

Solution. Adding $[ABE]$ to both sides of $[AED] = [BEC]$ gives $[ABD] = [ABC]$. These triangles have the common base AB, and C and D lie on the same side of line AB because $ABCD$ is convex. By Theorem 1.2.1, $DC \parallel AB$, so $ABCD$ is a trapezoid with bases AB and DC. By Theorem 2.2.1, $AE : EC = AB : DC = 9 : 12 = 3 : 4$, so

$$AE = \frac{3}{7} \cdot 14 = 6.$$

Answer: (E) 6. □

14. (2001 AMC 12, Problem #22) In rectangle $ABCD$, points F and G lie on $\overline{AB}$ so that $AF = FG = GB$ and E is the midpoint of $\overline{DC}$. Also, $\overline{AC}$ intersects $\overline{EF}$ at H and $\overline{EG}$ at J. The area of rectangle $ABCD$ is 70. Find the area of triangle EHJ.



(A) $\frac{5}{2}$ **(B)** $\frac{35}{12}$ **(C)** 3 **(D)** $\frac{7}{2}$ **(E)** $\frac{35}{8}$

Strategy. The segments AF and AG on AB are parallel to EC on DC. Theorem 2.2.1 of the handout, in the trapezoids $AFCE$ and $AGCE$, locates H and J on AC; then compare triangle EHJ with triangle EAC.

Solution. Let $AB = DC = 3w$. Then $AF = w$, $AG = 2w$ and $EC = \frac{3}{2}w$. The quadrilateral $AFCE$ is a trapezoid with parallel sides AF and EC, and its diagonals AC and FE meet at H. By Theorem 2.2.1,

$$\frac{AH}{HC} = \frac{AF}{EC} = \frac{w}{3w/2} = \frac{2}{3}, \qquad \text{so} \qquad AH = \tfrac{2}{5}\,AC.$$

In the same way, in trapezoid $AGCE$ the diagonals AC and GE meet at J, and

$$\frac{AJ}{JC} = \frac{AG}{EC} = \frac{2w}{3w/2} = \frac{4}{3}, \qquad \text{so} \qquad AJ = \tfrac{4}{7}\,AC.$$

So $HJ = \left(\frac{4}{7} - \frac{2}{5}\right)AC = \frac{6}{35}AC$. Triangles EHJ and EAC have the common vertex E and their bases HJ and AC on line AC, so $[EHJ] = \frac{6}{35}[EAC]$ by the equal-heights theorem. Triangle EAC has base $EC = \frac{1}{2}DC$ and height AD, so $[EAC] = \frac{1}{2} \cdot \frac{1}{2}DC \cdot AD = \frac{1}{4} \cdot 70 = \frac{35}{2}$. Therefore

$$[EHJ] = \frac{6}{35} \cdot \frac{35}{2} = 3.$$

Answer: (C) 3. □

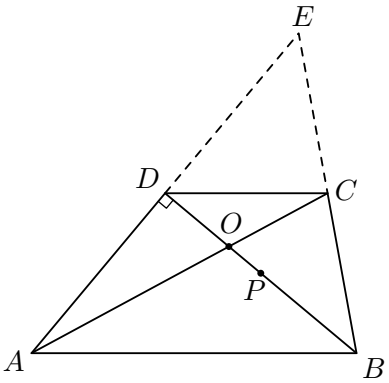
15. (2021 AMC 10A, Problem #17) Trapezoid $ABCD$ has $\overline{AB} \parallel \overline{CD}$, $BC = CD = 43$, and $\overline{AD} \perp \overline{BD}$. Let O be the intersection of the diagonals $\overline{AC}$ and $\overline{BD}$, and let P be the midpoint of $\overline{BD}$. Given that $OP = 11$, the length of AD can be written in the form $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. What is $m + n$?

(A) 65 **(B)** 132 **(C)** 157 **(D)** 194 **(E)** 215

Strategy. Show that $AB = 2\,CD$ by reflecting A across line BD. Then Theorem 2.2.1 of the handout places O on BD, and OP gives BD.

Solution. Since $AB \parallel CD$, the alternate interior angles $\angle ABD$ and $\angle BDC$ are equal, and since $CB = CD$, $\angle BDC = \angle DBC$. So $\angle ABD = \angle DBC$.

Reflect A across line BD to a point E. Since $AD \perp BD$, the point E lies on line AD with $DE = DA$; also $BE = BA$ and $\angle DBE = \angle DBA = \angle DBC$. The points E and C lie on the same side of line BD, the side away from A (the diagonals AC and BD cross, so A and C are on opposite sides of BD), so E lies on ray BC, as in the figure.



In triangle ABE, the point D is the midpoint of AE and $DC \parallel AB$, so by Theorem 2.1.1 the point C is the midpoint of BE and $DC = \frac{1}{2}AB$. Hence $AB = 86$.

By Theorem 2.2.1, $BO : OD = AB : DC = 2 : 1$, so $OD = \frac{1}{3}BD$, while $PD = \frac{1}{2}BD$. Therefore $OP = \frac{1}{2}BD - \frac{1}{3}BD = \frac{1}{6}BD = 11$, and $BD = 66$. Triangle ABD has a right angle at D, so

$$\begin{aligned} AD^2 &= AB^2 - BD^2 = 86^2 - 66^2 = (86-66)(86+66) \\ &= 20 \cdot 152 = 3040 = 16 \cdot 190, \end{aligned}$$

and $AD = 4\sqrt{190}$. Since $190 = 2 \cdot 5 \cdot 19$ is not divisible by the square of a prime, $m = 4$ and $n = 190$, so $m + n = 194$.

Answer: (D) 194. □

2. Answer Key

Problem	Letter	Value
1	D	The ratio of the altitude to the corresponding base
2	D	4
3	A	$3\frac{2}{3}$
4	B	$1:9$
5	C	$\frac{135}{2}$
6	C	200
7	D	$\frac{25}{2}$
8	E	20
9	C	5
10	E	$\frac{7}{16}$
11	C	$\frac{125}{12}$
12	B	$r = \frac{1}{2}$ independent of the position of P
13	E	6
14	C	3
15	D	194