TRIANGLE CENTERS, MEDIANS, AND CEVIANS FOR AMC 10

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1. Cevians and Areas

Medians, angle bisectors and altitudes are segments from a vertex of a triangle to the line through the other two vertices. This section names these segments and relates them to areas.

A **cevian** of triangle ABC is a segment AD from the vertex A to a point D on line BC, and D is its **foot**. Cevians from B and from C are defined in the same way. Unless stated otherwise, the foot lies on side BC. Three cevians from A have names of their own:

- the **median**, when D is the midpoint of BC;
- the **angle bisector**, when $\angle BAD = \angle DAC$;
- the **altitude**, when $AD \perp BC$.

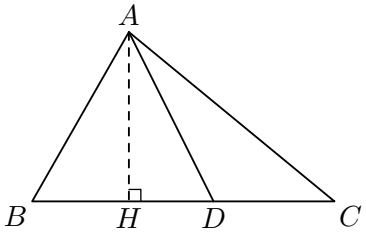
The foot of the altitude from A lies on side BC when the angles at B and C are both acute. If the foot H were on the extension of BC beyond B, then $\angle ABC$ would be the supplement of the acute angle $\angle ABH$ of the right triangle ABH, so it would be obtuse; the same holds at C. When the angle at B is obtuse, the foot lies on the extension of BC beyond B, and the altitude lies outside the triangle.

Throughout, $[ABC]$ is the area of triangle ABC and $[ABCD]$ is the area of quadrilateral $ABCD$. In triangle ABC the side lengths are $a = BC$, $b = CA$ and $c = AB$, and $K = [ABC]$. The **height** to side BC is the distance from A to line BC, which is the length of the altitude from A; if it is h, then $K = \frac{1}{2} \cdot BC \cdot h$.

1.1. Triangles with the Same Height.

Theorem 1.1.1 (Equal heights)**.** *Two triangles with equal heights have areas in the ratio of their bases. In particular, if D is a point of line BC, then*

$$[ABD] : [ACD] = BD : DC.$$



Proof. The area of each triangle is half its base times its height, and the heights are equal. For the second statement, triangles ABD and ACD both have the height $h = AH$ from A to line BC, dashed in the figure, so $[ABD] = \frac{1}{2} \cdot BD \cdot h$ and $[ACD] = \frac{1}{2} \cdot DC \cdot h$. □

Corollary 1.1.2. *A median divides a triangle into two triangles of equal area.*

Proof. If D is the midpoint of BC, then $BD = DC$, so $[ABD] = [ACD]$ by Theorem 1.1.1. □

The theorem is used most often for two triangles that share a vertex and have their bases on one line. The bases can be parts of a side or parts of another cevian.

Example.

Problem 1.1.3 (1976 AHSME, Problem #9)**.** In triangle ABC, D is the midpoint of AB; E is the midpoint of DB; and F is the midpoint of BC. If the area of $\triangle ABC$ is 96, then the area of $\triangle AEF$ is

(A) 16 **(B)** 24 **(C)** 32 **(D)** 36 **(E)** 48

Strategy. The segment AF is a median of triangle ABC, which gives $[ABF]$. Triangles AEF and ABF share the vertex F, and their bases AE and AB lie on one line.

Solution. Since F is the midpoint of BC, the median AF divides triangle ABC into two triangles of equal area, so $[ABF] = \frac{1}{2} \cdot 96 = 48$.

Since D is the midpoint of AB, $AD = \frac{1}{2}AB$, and since E is the midpoint of DB, $DE = \frac{1}{4}AB$. So $AE = \frac{3}{4}AB$. Triangles AEF and ABF have the same height from F to line AB, so by Theorem 1.1.1,

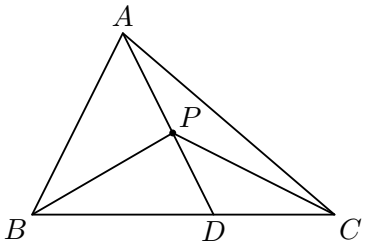
$$[AEF] = \frac{AE}{AB} \cdot [ABF] = \frac{3}{4} \cdot 48 = 36.$$

Answer: (D) 36. □

1.2. **Intersecting Cevians.** When cevians cross, Theorem 1.1.1 links the small triangles around the crossing point. The next theorem turns their areas into the ratio in which a cevian divides the opposite side.

Theorem 1.2.1 (A point on a cevian)**.** *Let AD be a cevian of triangle ABC, and let P be a point of segment AD other than A. Then*

$$[ABP] : [ACP] = BD : DC.$$

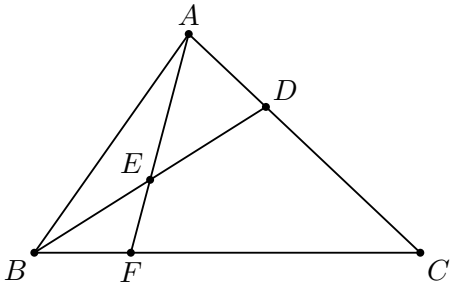


Proof. Let $k = BD/DC$. Triangles ABD and ACD have the common vertex A and their bases on line BC, so $[ABD] = k\,[ACD]$ by Theorem 1.1.1. In the same way, with the common vertex P, $[PBD] = k\,[PCD]$. Subtracting,

$$[ABP] = [ABD] - [PBD] = k\big([ACD] - [PCD]\big) = k\,[ACP].$$ □

Examples.

Problem 1.2.2 (2019 AMC 8, Problem #24)**.** In triangle ABC, point D divides side $\overline{AC}$ so that $AD : DC = 1 : 2$. Let E be the midpoint of $\overline{BD}$ and let F be the point of intersection of line BC and line AE. Given that the area of $\triangle ABC$ is 360, what is the area of $\triangle EBF$?



(A) 24 **(B)** 30 **(C)** 32 **(D)** 36 **(E)** 40

Strategy. Find the areas of the four triangles around E with Theorem 1.1.1. Then Theorem 1.2.1, applied to the cevian AF through E, gives the ratio $BF : FC$.

Solution. Triangles ABD and CBD have the common vertex B and their bases on line AC, with $AD : DC = 1 : 2$, so $[ABD] = 120$ and $[CBD] = 240$. The point E is the midpoint of BD, so AE is a median of triangle ABD and CE is a median of triangle CBD. By Corollary 1.1.2,

$$[ABE] = [ADE] = 60 \quad \text{and} \quad [CBE] = [CDE] = 120.$$

The point E lies on the cevian AF of triangle ABC, so by Theorem 1.2.1,

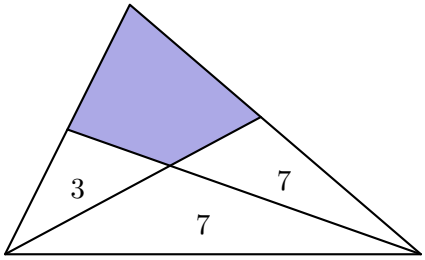
$$BF : FC = [ABE] : [ACE] = 60 : (60 + 120) = 1 : 3.$$

So $BF = \frac{1}{4}BC$. Triangles EBF and EBC have the same height from E to line BC, so

$$[EBF] = \tfrac{1}{4}\,[EBC] = \tfrac{1}{4} \cdot 120 = 30.$$

Answer: (B) 30. □

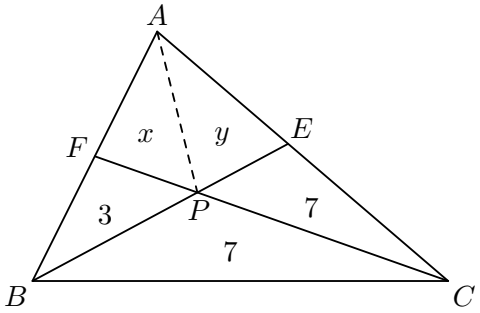
Problem 1.2.3 (2006 AMC 10B, Problem #23)**.** A triangle is partitioned into three triangles and a quadrilateral by drawing two lines from vertices to their opposite sides. The areas of the three triangles are 3, 7, and 7, as shown. What is the area of the shaded quadrilateral?



(A) 15 **(B)** 17 **(C)** $\frac{35}{2}$ **(D)** 18 **(E)** $\frac{55}{3}$

Strategy. Name the points, and join the top vertex to the crossing point; call the areas of the two pieces of the shaded region x and y. Theorem 1.1.1, applied along each of the two drawn lines, gives two equations.

Solution. Name the points as in the figure below. The triangle is ABC with base BC, the drawn lines are BE and CF with E on CA and F on AB, and they meet at P. Then $[BFP] = 3$, $[BPC] = 7$ and $[CEP] = 7$. Draw AP, and let $x = [AFP]$ and $y = [AEP]$.



The points F, P, C lie on a line. Triangles BFP and BPC have the common vertex B, so $FP : PC = [BFP] : [BPC] = 3 : 7$. Triangles AFP and APC have the common vertex A, so $FP : PC = [AFP] : [APC] = x : (y + 7)$. Hence

$$\frac{x}{y+7} = \frac{3}{7}, \qquad \text{so} \qquad 7x = 3y + 21.$$

In the same way, the points E, P, B lie on a line. Triangles CEP and CPB have the common vertex C, so $EP : PB = 7 : 7 = 1 : 1$, and triangles AEP and APB have the common vertex A, so $EP : PB = y : (x + 3)$. Hence $y = x + 3$. Substituting,

$$7x = 3(x + 3) + 21, \qquad 4x = 30,$$

so $x = \frac{15}{2}$ and $y = \frac{21}{2}$. The area of the quadrilateral is $x + y = 18$. (The figure is not drawn to scale.)

Answer: (D) 18. □

2. Medians and the Centroid

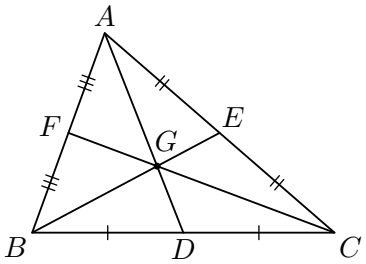
The three medians of a triangle pass through one point. Its position on each median, and the areas of the six pieces that the medians cut, follow from Section 1.

2.1. **The Centroid.**

Theorem 2.1.1 (Centroid)**.** *The three medians AD, BE and CF of triangle ABC meet at one point G, the **centroid**. It divides each median in the ratio $2:1$ from the vertex:*

$$AG = 2\,GD, \qquad BG = 2\,GE, \qquad CG = 2\,GF.$$

The medians cut the triangle into six triangles of equal area $\frac{K}{6}$. In particular, $[GBC] = [GCA] = [GAB] = \frac{K}{3}$.



Proof. Let the medians BE and CF meet at G. Since E is the midpoint of CA, Corollary 1.1.2 gives $[ABE] = [CBE]$, from the median BE of triangle ABC, and $[AGE] = [CGE]$, from the median GE of triangle GCA. Subtracting, $[ABG] = [CBG]$. In the same way, the midpoint F of AB gives $[CAG] = [CBG]$. So the triangles GAB, GBC and GCA have equal areas, and since together they make up triangle ABC, each has area $\frac{K}{3}$.

Let line AG meet BC at D. By Theorem 1.2.1, $BD : DC = [ABG] : [ACG] = 1$, so D is the midpoint of BC, and the third median AD passes through G as well.

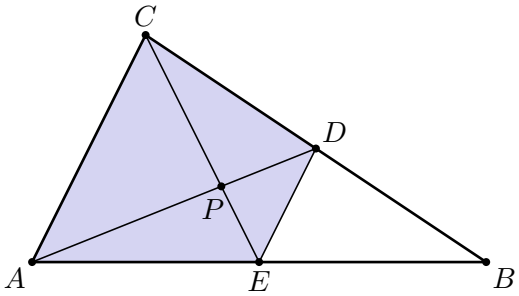
Each of the triangles GBC, GCA and GAB is split by GD, GE or GF into two triangles of equal area, by Corollary 1.1.2, so all six small triangles have area $\frac{K}{6}$. Finally, triangles ABG and DBG have the same height from B to line AD, so by Theorem 1.1.1,

$$AG : GD = [ABG] : [DBG] = \tfrac{K}{3} : \tfrac{K}{6} = 2 : 1.$$

The other two medians are divided in the same way. □

Examples.

Problem 2.1.2 (2013 AMC 10B, Problem #16)**.** In triangle $\triangle ABC$, medians AD and CE intersect at P, $PE = 1.5$, $PD = 2$, and $DE = 2.5$. What is the area of $AEDC$?



(A) 13 **(B)** 13.5 **(C)** 14 **(D)** 14.5 **(E)** 15

Strategy. The point P is the centroid, so it divides each median in the ratio $2:1$. The three given lengths form a right triangle, so the diagonals of $AEDC$ are perpendicular.

Solution. The medians AD and CE meet at the centroid, so P is the centroid of triangle ABC. By Theorem 2.1.1,

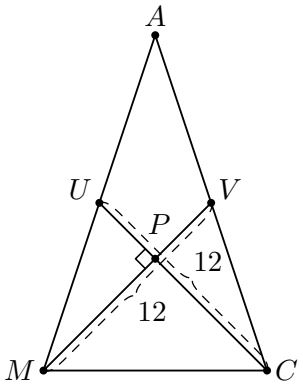
$$AP = 2\,PD = 4 \quad \text{and} \quad CP = 2\,PE = 3.$$

Since $1.5^2 + 2^2 = 2.25 + 4 = 6.25 = 2.5^2$, the converse of the Pythagorean theorem shows that triangle EPD has a right angle at P. So the diagonals AD and CE of quadrilateral $AEDC$ are perpendicular, and they cut it into four right triangles whose legs lie along the diagonals:

$$[AEDC] = \tfrac{1}{2} \cdot AP \cdot PE + \tfrac{1}{2} \cdot PE \cdot PD + \tfrac{1}{2} \cdot PD \cdot PC + \tfrac{1}{2} \cdot PC \cdot PA = 3 + 1.5 + 3 + 6 = 13.5.$$

Answer: (B) 13.5. □

Problem 2.1.3 (2020 AMC 10A, Problem #12)**.** Triangle AMC is isosceles with $AM = AC$. Medians $\overline{MV}$ and $\overline{CU}$ are perpendicular to each other, and $MV = CU = 12$. What is the area of $\triangle AMC$?



(A) 48 **(B)** 72 **(C)** 96 **(D)** 144 **(E)** 192

Strategy. The triangle with vertices M, C and the centroid is a right triangle with known legs, and its area is one third of the area of triangle AMC.

Solution. The medians MV and CU meet at P, the centroid of triangle AMC. By Theorem 2.1.1,

$$MP = \tfrac{2}{3} \cdot MV = 8 \quad \text{and} \quad CP = \tfrac{2}{3} \cdot CU = 8.$$

The medians are perpendicular, so triangle PMC has a right angle at P, and $[PMC] = \frac{1}{2} \cdot 8 \cdot 8 = 32$. By Theorem 2.1.1, $[PMC] = \frac{1}{3}[AMC]$, so

$$[AMC] = 3 \cdot 32 = 96.$$

The condition $AM = AC$ is not used: the lengths of the two medians and the right angle between them already determine the area.

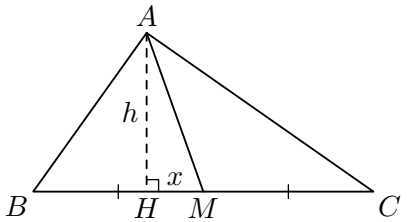
Answer: (C) 96. □

2.2. **The Length of a Median.**

Theorem 2.2.1 (Median length)**.** *If M is the midpoint of side BC of triangle ABC, then*

$$AB^2 + AC^2 = 2AM^2 + 2BM^2.$$

Equivalently, $AM^2 = \frac{1}{4}\left(2b^2 + 2c^2 - a^2\right)$.



Proof. Let H be the foot of the altitude from A, and let $h = AH$. Suppose first that H lies between B and M, as in the figure, and let $x = MH$. Then $BH = \frac{a}{2} - x$ and $CH = \frac{a}{2} + x$, and the Pythagorean theorem in the right triangles ABH, ACH and AMH gives

$$c^2 = h^2 + \left(\frac{a}{2} - x\right)^2, \qquad b^2 = h^2 + \left(\frac{a}{2} + x\right)^2, \qquad AM^2 = h^2 + x^2.$$

Adding the first two equations,

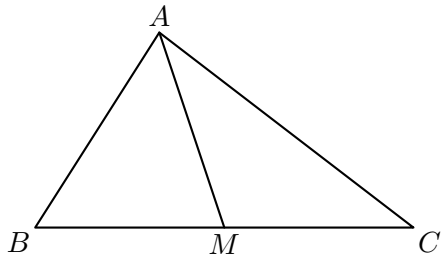
$$b^2 + c^2 = 2h^2 + 2x^2 + 2\left(\frac{a}{2}\right)^2 = 2AM^2 + 2BM^2.$$

In every other position of H on line BC, including positions outside segment BC, the same three equations hold with $x = MH$, except that $\frac{a}{2} - x$ and $\frac{a}{2} + x$ may be exchanged, so the sum is the same. The second form follows from $BM = \frac{a}{2}$. □

When the angle at C is right, the median from A is the hypotenuse of the right triangle ACM, so the Pythagorean theorem gives $AM^2 = AC^2 + CM^2$ directly.

Example.

Problem 2.2.2 (1975 AHSME, Problem #20)**.** In the adjoining figure triangle ABC is such that $AB = 4$ and $AC = 8$. If M is the midpoint of BC and $AM = 3$, what is the length of BC?



(A) $2\sqrt{26}$ **(B)** $2\sqrt{31}$ **(C)** 9 **(D)** $4 + 2\sqrt{13}$
(E) not enough information given to solve the problem

Strategy. Use Theorem 2.2.1 with $BM = \frac{1}{2}BC$.

Solution. By Theorem 2.2.1,

$$AB^2 + AC^2 = 2AM^2 + 2BM^2, \qquad \text{so} \qquad 16 + 64 = 18 + 2BM^2.$$

Then $BM^2 = 31$, $BM = \sqrt{31}$ and $BC = 2\,BM = 2\sqrt{31}$. A triangle with sides 4, 8 and $2\sqrt{31}$ exists, because $4 + 8 = 12 > 2\sqrt{31}$. The figure is not drawn to scale.

Answer: (B) $2\sqrt{31}$. □

3. Angle Bisectors and the Incenter

An angle bisector divides the opposite side in the ratio of the two sides at its vertex. The three angle bisectors meet at the center of the circle that touches all three sides.

3.1. **The Angle Bisector Theorem.**

Theorem 3.1.1 (Points on a bisector)**.** *Let P be a point inside $\angle BAC$. Then P lies on the bisector of $\angle BAC$ if and only if P is equally far from the lines AB and AC. In that case the feet X and Y of the perpendiculars from P to AB and AC lie on the rays AB and AC, and $AX = AY$.*

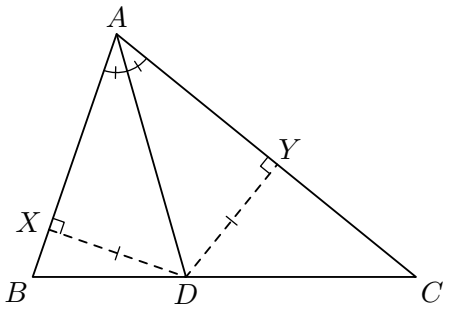
Proof. Suppose P lies on the bisector. The angles $\angle PAB$ and $\angle PAC$ are each half of $\angle BAC$, so they are acute, and X and Y lie on the rays AB and AC. The right triangles AXP and AYP have the common hypotenuse AP and equal angles at A, so they are congruent (AAS). Hence $PX = PY$ and $AX = AY$.

Conversely, suppose P is at the same distance d from both lines. The points on the same side of line AB as P and at distance d from it form the line ℓ through P parallel to AB. In the same way, the points on the same side of line AC as P and at distance d from it form the line m through P parallel to AC. The line ℓ crosses the bisector of $\angle BAC$ at a point Q inside the angle. By the first part, Q is also at distance d from AC, so Q lies on m. The lines ℓ and m are not parallel, so they meet only at P. Hence $Q = P$, and P lies on the bisector. □

Theorem 3.1.2 (Angle bisector theorem)**.** *If D lies on side BC of triangle ABC and AD bisects $\angle BAC$, then*

$$BD : DC = AB : AC.$$

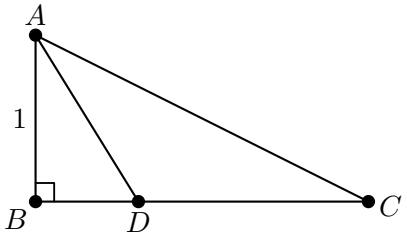
Consequently, $BD = \dfrac{ac}{b+c}$ *and* $DC = \dfrac{ab}{b+c}$.



Proof. By Theorem 1.1.1, $[ABD] : [ACD] = BD : DC$. Since D lies on the bisector of $\angle BAC$, Theorem 3.1.1 shows that D is at the same distance d from the lines AB and AC; these are the dashed segments DX and DY in the figure. Taking AB and AC as bases, $[ABD] = \frac{1}{2} \cdot AB \cdot d$ and $[ACD] = \frac{1}{2} \cdot AC \cdot d$, so $[ABD] : [ACD] = AB : AC$. This proves $BD : DC = c : b$. Since $BD + DC = a$, it follows that $BD = \frac{c}{b+c} \cdot a$ and $DC = \frac{b}{b+c} \cdot a$. □

Examples.

Problem 3.1.3 (2009 AMC 10B, Problem #20)**.** Triangle ABC has a right angle at B, $AB = 1$, and $BC = 2$. The bisector of $\angle BAC$ meets $\overline{BC}$ at D. What is BD?



(A) $\dfrac{\sqrt{3}-1}{2}$ **(B)** $\dfrac{\sqrt{5}-1}{2}$ **(C)** $\dfrac{\sqrt{5}+1}{2}$ **(D)** $\dfrac{\sqrt{6}+\sqrt{2}}{2}$ **(E)** $2\sqrt{3}-1$

Strategy. Find AC with the Pythagorean theorem. Then divide BC in the ratio $AB : AC$.

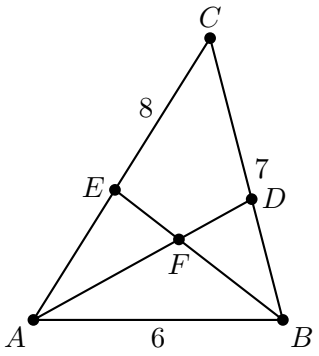
Solution. The hypotenuse is $AC = \sqrt{AB^2 + BC^2} = \sqrt{5}$, and Theorem 3.1.2 with $a = BC = 2$, $b = CA = \sqrt{5}$ and $c = AB = 1$ gives

$$BD = \frac{ac}{b+c} = \frac{2}{\sqrt{5}+1} = \frac{2(\sqrt{5}-1)}{(\sqrt{5}+1)(\sqrt{5}-1)} = \frac{\sqrt{5}-1}{2},$$

since $(\sqrt{5}+1)(\sqrt{5}-1) = 5-1 = 4$.

Answer: (B) $\dfrac{\sqrt{5}-1}{2}$. □

Problem 3.1.4 (2016 AMC 12A, Problem #12)**.** In $\triangle ABC$, $AB = 6$, $BC = 7$, and $CA = 8$. Point D lies on $\overline{BC}$, and $\overline{AD}$ bisects $\angle BAC$. Point E lies on $\overline{AC}$, and $\overline{BE}$ bisects $\angle ABC$. The bisectors intersect at F. What is the ratio $AF : FD$?



(A) 3 : 2 **(B)** 5 : 3 **(C)** 2 : 1 **(D)** 7 : 3 **(E)** 5 : 2

Strategy. Use Theorem 3.1.2 twice: for the bisector from A in triangle ABC, which places D, and for the bisector from B in triangle ABD, which divides AD at F.

Solution. By the angle bisector theorem in triangle ABC,

$$BD = \frac{BC \cdot AB}{AB + AC} = \frac{7 \cdot 6}{6 + 8} = 3.$$

Since D lies on side BC, $\angle ABD = \angle ABC$, so BF bisects the angle ABD of triangle ABD and meets the opposite side AD at F. By the angle bisector theorem in triangle ABD,

$$AF : FD = BA : BD = 6 : 3 = 2 : 1.$$

The point F is the incenter of triangle ABC, the subject of Section 3.2.

Answer: (C) 2 : 1. □

3.2. **The Incenter and the Inradius.**

Theorem 3.2.1 (Incenter)**.** *The three angle bisectors of a triangle meet at one point I, the* ***incenter****, which is at the same distance r from all three sides. The circle with center I and radius r touches each side at the foot of the perpendicular from I to that side. It is the* ***inscribed circle****, or* ***incircle****, and r is the* ***inradius****.*

Proof. The bisectors of $\angle B$ and $\angle C$ meet at a point I inside the triangle. By Theorem 3.1.1, I is equally far from AB and BC, and equally far from BC and CA. So I is equally far from AB and AC, and by the converse part of Theorem 3.1.1 it lies on the bisector of $\angle A$. Let r be the common distance. By the last part of Theorem 3.1.1, applied at B and at C, the foot of the perpendicular from I to BC lies on both rays BC and CB, so it lies on side BC; the same holds for the other two sides. So the circle with center I and radius r touches each side at that foot. □

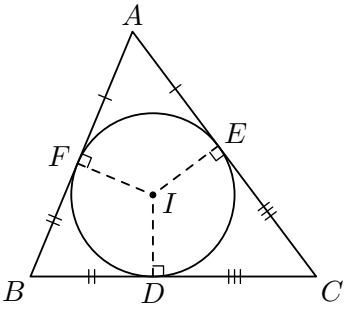
Theorem 3.2.2 (Area and inradius)**.** $K = rs$*, where* $s = \frac{a+b+c}{2}$ *is the* ***semiperimeter****.*

Proof. The segments IA, IB and IC cut triangle ABC into the triangles IBC, ICA and IAB, with bases a, b, c and the common height r. So

$$K = \tfrac{1}{2}ar + \tfrac{1}{2}br + \tfrac{1}{2}cr = \tfrac{1}{2}(a+b+c)\,r = rs.$$ □

Theorem 3.2.3 (Tangent lengths)**.** *If the incircle touches BC, CA and AB at D, E and F, then*

$$AE = AF = s - a, \qquad BF = BD = s - b, \qquad CD = CE = s - c.$$



Proof. The incenter I lies on all three angle bisectors, so by the last part of Theorem 3.1.1, applied at each vertex, $AE = AF = x$, $BF = BD = y$ and $CD = CE = z$ for some numbers x, y, z. The touch points lie on the sides, so

$$y + z = a, \qquad z + x = b, \qquad x + y = c.$$

Adding, $2(x + y + z) = a + b + c$, so $x + y + z = s$. Subtracting each of the three equations from this gives $x = s - a$, $y = s - b$ and $z = s - c$. □

Corollary 3.2.4 (Right triangles)**.** *If $\angle C = 90°$, then $r = s - c = \dfrac{a + b - c}{2}$.*

Proof. With D and E as in Theorem 3.2.3, the quadrilateral $IDCE$ has right angles at D, C and E, so it is a rectangle, and $ID = IE = r$, so it is a square. Hence $r = CD = s - c$, and $s - c = \frac{a+b+c}{2} - c = \frac{a+b-c}{2}$. □

Examples.

Problem 3.2.5 (2007 AMC 10B, Problem #17)**.** Point P is inside equilateral $\triangle ABC$. Points Q, R, and S are the feet of the perpendiculars from P to $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$, respectively. Given that $PQ = 1$, $PR = 2$, and $PS = 3$, what is AB?

(A) 4 **(B)** $3\sqrt{3}$ **(C)** 6 **(D)** $4\sqrt{3}$ **(E)** 9

Strategy. Split the triangle into three triangles with the common vertex P, as in the proof of Theorem 3.2.2. Their heights are the given perpendiculars.

Solution. Let $x = AB = BC = CA$. The segments PA, PB and PC cut triangle ABC into the triangles PAB, PBC and PCA, whose heights from P are $PQ = 1$, $PR = 2$ and $PS = 3$. So

$$[ABC] = \tfrac{1}{2} \cdot x \cdot 1 + \tfrac{1}{2} \cdot x \cdot 2 + \tfrac{1}{2} \cdot x \cdot 3 = 3x.$$

The area of an equilateral triangle with side x is $\frac{\sqrt{3}}{4}x^2$. Hence $\frac{\sqrt{3}}{4}x^2 = 3x$, and

$$x = \frac{12}{\sqrt{3}} = 4\sqrt{3}.$$

The same argument shows that the three distances from any point inside an equilateral triangle to its sides add to the height of the triangle, here $1 + 2 + 3 = 6$.

Answer: (D) $4\sqrt{3}$. □

Problem 3.2.6 (1954 AHSME, Problem #43)**.** The hypotenuse of a right triangle is 10 inches and the radius of the inscribed circle is 1 inch. The perimeter of the triangle in inches is:

(A) 15 **(B)** 22 **(C)** 24 **(D)** 26 **(E)** 30

Strategy. Corollary 3.2.4 gives the sum of the legs directly.

Solution. Let the legs be a and b. By Corollary 3.2.4,

$$1 = r = \frac{a+b-10}{2},$$

so $a+b=12$, and the perimeter is $a+b+10=22$. Such a triangle exists: the legs satisfy $a+b=12$ and $a^2+b^2=100$, so $ab=\frac{144-100}{2}=22$, and the legs are the roots $6\pm\sqrt{14}$ of $t^2-12t+22=0$, both positive.

Answer: (B) 22. □

4. Perpendicular Bisectors and the Circumcenter

The points equally far from two given points form a line. For the three sides of a triangle these lines meet at one point, the center of the circle through the three vertices.

4.1. The Circumcenter.

The **perpendicular bisector** of a segment AB is the line through the midpoint of AB perpendicular to AB.

Theorem 4.1.1 (Perpendicular bisector)**.** *A point P satisfies $PA=PB$ if and only if P lies on the perpendicular bisector of AB.*

Proof. Let M be the midpoint of AB. If $P=M$, then $PA=PB$. If $P\neq M$ lies on the perpendicular bisector, the right triangles PMA and PMB give

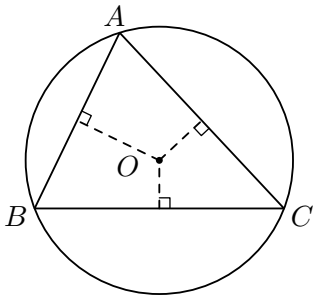
$$PA^2 = PM^2 + MA^2 = PM^2 + MB^2 = PB^2.$$

Conversely, suppose $PA=PB$. If P lies on line AB, then $P=M$. Otherwise, let H be the foot of the perpendicular from P to line AB. Then

$$HA^2 = PA^2 - PH^2 = PB^2 - PH^2 = HB^2,$$

so H is the point of line AB equally far from A and B, which is M. Hence $PM\perp AB$, and P lies on the perpendicular bisector. □

Theorem 4.1.2 (Circumcenter)**.** *The perpendicular bisectors of the three sides of a triangle meet at one point O, the **circumcenter**, and $OA=OB=OC$. The circle with center O and radius $R=OA$ passes through all three vertices. It is the **circumscribed circle**, or **circumcircle**, and R is the **circumradius**.*



Proof. The perpendicular bisectors of AB and AC are not parallel, because AB and AC are not, so they meet at a point O. By Theorem 4.1.1, $OA=OB$ and $OA=OC$. Then $OB=OC$, so O lies on the perpendicular bisector of BC as well. □

The circumcenter lies inside an acute triangle, at the midpoint of the hypotenuse of a right triangle (Section 4.2), and outside an obtuse triangle. If $AB=AC$, Theorem 4.1.1 puts A on the perpendicular bisector of BC. That line passes through A and is perpendicular to BC, so it is the line of the altitude from A. Hence the circumcenter of an isosceles triangle lies on the line of the altitude to its base.

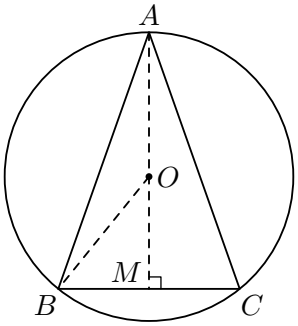
Example.

Problem 4.1.3 (2007 AMC 10B, Problem #11)**.** A circle passes through the three vertices of an isosceles triangle that has two sides of length 3 and a base of length 2. What is the area of this circle?

(A) 2π **(B)** $\frac{5}{2}\pi$ **(C)** $\frac{81}{32}\pi$ **(D)** 3π **(E)** $\frac{7}{2}\pi$

Strategy. The center lies on the line of the altitude to the base. Write $OB = OA = R$ with the Pythagorean theorem.

Solution. Let $AB = AC = 3$ and $BC = 2$, and let M be the midpoint of BC. The altitude AM has length $\sqrt{3^2 - 1^2} = 2\sqrt{2}$, and the center O of the circle, the circumcenter of triangle ABC, lies on line AM.



Let $R = OA = OB$. With O between A and M, as in the figure, $OM = 2\sqrt{2} - R$, and the right triangle OMB gives

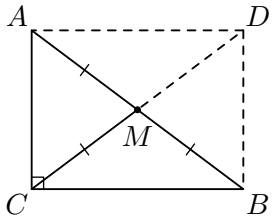
$$R^2 = OM^2 + MB^2 = \left(2\sqrt{2} - R\right)^2 + 1 = 9 - 4\sqrt{2}\,R + R^2.$$

So $4\sqrt{2}\,R = 9$ and $R = \frac{9}{4\sqrt{2}}$. This is less than $2\sqrt{2} = \frac{16}{4\sqrt{2}}$, so O does lie between A and M. Then $R^2 = \frac{81}{32}$, and the area of the circle is $\pi R^2 = \frac{81}{32}\pi$.

Answer: (C) $\frac{81}{32}\pi$. □

4.2. **Right Triangles.**

Theorem 4.2.1 (Midpoint of the hypotenuse)**.** *In a right triangle the circumcenter is the midpoint of the hypotenuse. Equivalently, the median to the hypotenuse is half as long as the hypotenuse, and $R = \frac{c}{2}$ when c is the hypotenuse.*



Proof. Let $\angle C = 90°$ and let M be the midpoint of the hypotenuse AB. Extend CM past M to the point D with $MD = CM$. The diagonals AB and CD of quadrilateral $ACBD$ bisect each other, so it is a parallelogram. It has a right angle at C, so it is a rectangle, and the diagonals of a rectangle are equal: $CD = AB$. Hence

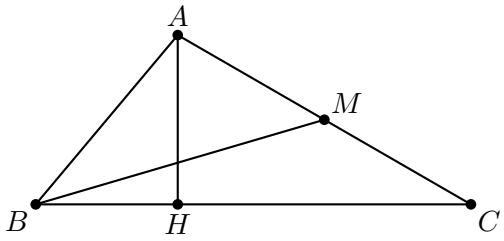
$$MC = \tfrac{1}{2}\,CD = \tfrac{1}{2}\,AB = MA = MB.$$

So M is equally far from A, B and C. By Theorem 4.1.1 it lies on the perpendicular bisectors of all three sides, so M is the circumcenter, and $R = MA = \frac{c}{2}$. □

An altitude of any triangle creates two right triangles, and the theorem applies to each of them.

Example.

Problem 4.2.2 (1989 AHSME, Problem #7)**.** In $\triangle ABC$, $\angle A = 100°$, $\angle B = 50°$, $\angle C = 30°$, AH is an altitude, and BM is a median. Then $\angle MHC =$



(A) 15° **(B)** 22.5° **(C)** 30° **(D)** 40° **(E)** 45°

Strategy. In the right triangle AHC, the segment HM is the median to the hypotenuse AC.

Solution. The angles at B and C are acute, so the foot H of the altitude from A lies on side BC (Section 1), and triangle AHC has a right angle at H and hypotenuse AC. The median BM makes M the midpoint of AC, so by Theorem 4.2.1, $MH = MC$. Triangle MHC is isosceles, so

$$\angle MHC = \angle MCH = \angle ACB = 30°.$$

The angle at A is not used, and the angle at B serves only to place H on side BC.

Answer: (C) 30°. □

5. Altitudes and the Orthocenter

An altitude and its base give the area of the triangle, which links the three altitudes to the three sides. The lines containing the three altitudes also meet at one point.

5.1. **Altitudes and Area.** Let h_a, h_b and h_c be the lengths of the altitudes to the sides BC, CA and AB.

Theorem 5.1.1 (Altitudes and sides)**.**

$$h_a = \frac{2K}{a}, \qquad h_b = \frac{2K}{b}, \qquad h_c = \frac{2K}{c}.$$

Hence $h_a : h_b : h_c = \dfrac{1}{a} : \dfrac{1}{b} : \dfrac{1}{c}$*, and the shortest altitude is the one to the longest side.*

Proof. Each side can serve as the base: $K = \frac{1}{2}ah_a = \frac{1}{2}bh_b = \frac{1}{2}ch_c$. □

Examples.

Problem 5.1.2 (2013 AMC 10A, Problem #15)**.** Two sides of a triangle have lengths 10 and 15. The length of the altitude to the third side is the average of the lengths of the altitudes to the two given sides. How long is the third side?

(A) 6 **(B)** 8 **(C)** 9 **(D)** 12 **(E)** 18

Strategy. Write every altitude as $2K$ divided by its side. Then K cancels.

Solution. Let the third side be x and the area K. By Theorem 5.1.1, the altitudes to the sides 10, 15 and x are $\frac{2K}{10}$, $\frac{2K}{15}$ and $\frac{2K}{x}$, so

$$\frac{2K}{x} = \frac{1}{2}\left(\frac{2K}{10} + \frac{2K}{15}\right).$$

Dividing by $2K$,

$$\frac{1}{x} = \frac{1}{2}\left(\frac{1}{10} + \frac{1}{15}\right) = \frac{1}{2}\cdot\frac{1}{6} = \frac{1}{12},$$

so $x = 12$. The lengths 10, 15 and 12 do form a triangle.

Answer: (D) 12. □

Problem 5.1.3 (2009 AMC 12B, Problem #13)**.** Triangle ABC has $AB = 13$ and $AC = 15$, and the altitude to BC has length 12. What is the sum of the two possible values of BC?

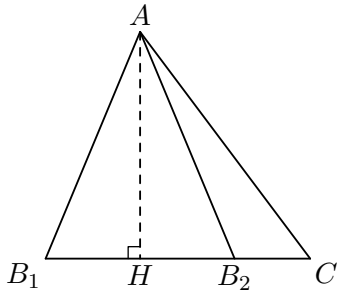
(A) 15 **(B)** 16 **(C)** 17 **(D)** 18 **(E)** 19

Strategy. The foot of the altitude can fall between B and C or outside segment BC. Find its distances from B and C with the Pythagorean theorem.

Solution. The altitude from A meets line BC at a point H with $AH = 12$, and the right triangles AHB and AHC give

$$HB = \sqrt{AB^2 - AH^2} = \sqrt{169 - 144} = 5 \quad \text{and} \quad HC = \sqrt{AC^2 - AH^2} = \sqrt{225 - 144} = 9.$$

The figure marks the two points B_1 and B_2 of line CH at distance 5 from H.

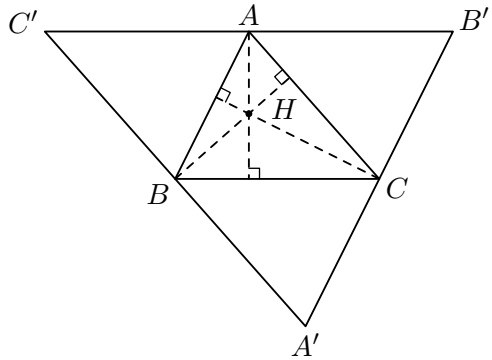


If B is B_1, on the other side of H from C, then $BC = HB + HC = 14$. If B is B_2, between H and C (as $HB < HC$), then the angle ABC is obtuse (Section 1) and $BC = HC - HB = 4$. Both triangles exist, so the sum of the two possible values is $14 + 4 = 18$.

Answer: (D) 18. □

5.2. **The Orthocenter.**

Theorem 5.2.1 (Orthocenter)**.** *The three lines containing the altitudes of a triangle meet at one point H, the* ***orthocenter****.*

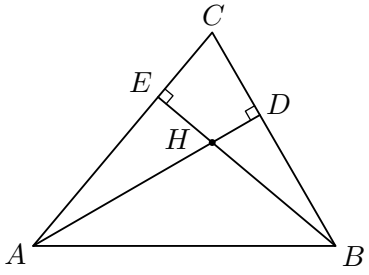


Proof. Through each vertex of triangle ABC draw the line parallel to the opposite side. These three lines form a larger triangle $A'B'C'$, with A' opposite A, B' opposite B and C' opposite C, as in the figure. The quadrilaterals $ABCB'$ and $ACBC'$ have both pairs of opposite sides parallel, so they are parallelograms, and therefore $AB' = BC = AC'$. So A is the midpoint of $B'C'$. The altitude from A is perpendicular to BC, hence to $B'C'$, and it passes through the midpoint A of $B'C'$: it is the perpendicular bisector of $B'C'$. In the same way, the altitudes from B and C are the perpendicular bisectors of $C'A'$ and $A'B'$. By Theorem 4.1.2, the three perpendicular bisectors of triangle $A'B'C'$ meet at one point. □

The orthocenter lies inside an acute triangle. In a right triangle the two legs are altitudes, so the orthocenter is the vertex of the right angle. In an obtuse triangle the orthocenter lies outside the triangle.

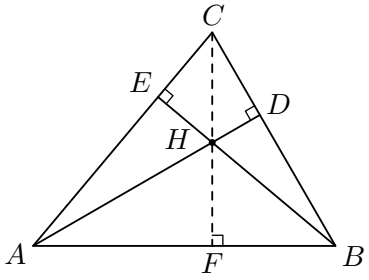
Example.

Problem 5.2.2. In triangle ABC, $\angle BAC = 50°$ and $\angle ABC = 60°$. The altitude from A meets BC at D, the altitude from B meets CA at E, and AD and BE meet at H. Find $\angle ACH$ and $\angle AHB$.



Strategy. By Theorem 5.2.1, line CH is the third altitude. For $\angle AHB$, use the quadrilateral $CDHE$, which has two right angles.

Solution. The third angle is $\angle ACB = 180° - 50° - 60° = 70°$, so all three angles are acute. By Theorem 5.2.1, line CH contains the altitude from C; let it meet AB at F, dashed in the figure.



The right triangle AFC has $\angle FAC = 50°$, so

$$\angle ACH = \angle ACF = 90° - 50° = 40°.$$

In quadrilateral $CDHE$ the angles at D and E are right angles and the angle at C is $70°$, so

$$\angle DHE = 360° - 90° - 90° - 70° = 110°.$$

The points A, H, D lie on a line, and so do B, H, E, so $\angle AHB$ and $\angle DHE$ are vertical angles. Hence $\angle AHB = 110°$. The same argument shows that $\angle AHB = 180° - \angle ACB$ in every acute triangle.

Answer: $\angle ACH = 40°$ and $\angle AHB = 110°$. □

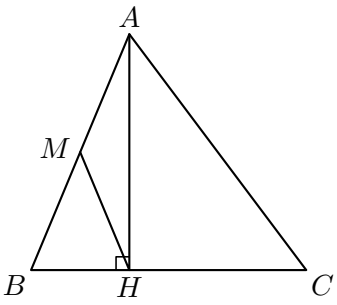
6. Problems

Beginner.

1. (2014 AMC 10A, Problem #9) The two legs of a right triangle, which are altitudes, have lengths $2\sqrt{3}$ and 6. How long is the third altitude of the triangle?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

2. (1986 AHSME, Problem #11) In $\triangle ABC$, $AB = 13$, $BC = 14$ and $CA = 15$. Also, M is the midpoint of side AB and H is the foot of the altitude from A to BC. The length of HM is



(A) 6 **(B)** 6.5 **(C)** 7 **(D)** 7.5 **(E)** 8

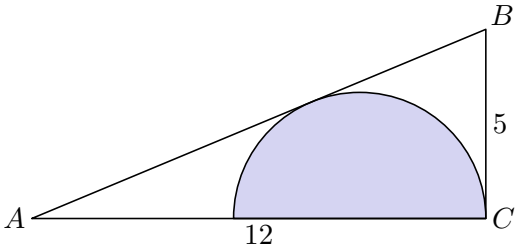
3. (1964 AHSME, Problem #38) The sides PQ and PR of triangle PQR are respectively of lengths 4 inches and 7 inches. The median PM is $3\frac{1}{2}$ inches. Then QR, in inches, is:

(A) 6 **(B)** 7 **(C)** 8 **(D)** 9 **(E)** 10

4. (1965 AHSME, Problem #16) Let line AC be perpendicular to line CE. Connect A to the midpoint D of CE, and connect E to the midpoint B of AC. If AD and EB intersect in point F, and $BC = CD = 15$ inches, then the area of triangle DFE, in square inches, is:

(A) 50 **(B)** $50\sqrt{2}$ **(C)** 75 **(D)** $\frac{15}{2}\sqrt{105}$ **(E)** 100

5. (2017 AMC 8, Problem #22) In the right triangle ABC, $AC = 12$, $BC = 5$, and angle C is a right angle. A semicircle is inscribed in the triangle as shown. What is the radius of the semicircle?



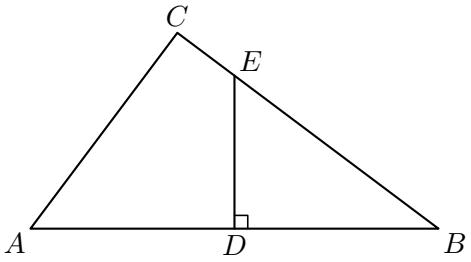
(A) $\frac{7}{6}$ **(B)** $\frac{13}{5}$ **(C)** $\frac{59}{18}$ **(D)** $\frac{10}{3}$ **(E)** $\frac{60}{13}$

Intermediate.

6. (2002 AMC 10B, Problem #22) Let $\triangle XOY$ be a right-angled triangle with $m\angle XOY = 90^\circ$. Let M and N be the midpoints of legs OX and OY, respectively. Given that $XN = 19$ and $YM = 22$, what is XY?

(A) 24 **(B)** 26 **(C)** 28 **(D)** 30 **(E)** 32

7. (1952 AHSME, Problem #24) In the figure, it is given that angle $C = 90°$, $AD = DB$, $DE \perp AB$, $AB = 20$, and $AC = 12$. The area of quadrilateral $ADEC$ is:



(A) 75 **(B)** $58\frac{1}{2}$ **(C)** 48 **(D)** $37\frac{1}{2}$ **(E)** None of these

8. (2000 AMC 12, Problem #19) In triangle ABC, $AB = 13$, $BC = 14$, and $AC = 15$. Let D denote the midpoint of $\overline{BC}$ and let E denote the intersection of $\overline{BC}$ with the bisector of angle BAC. Which of the following is closest to the area of triangle ADE?

(A) 2 **(B)** 2.5 **(C)** 3 **(D)** 3.5 **(E)** 4

9. (1973 AHSME, Problem #9) In $\triangle ABC$ with right angle at C, altitude CH and median CM trisect the right angle. If the area of $\triangle CHM$ is K, then the area of $\triangle ABC$ is

(A) $6K$ **(B)** $4\sqrt{3}K$ **(C)** $3\sqrt{3}K$ **(D)** $3K$ **(E)** $4K$

10. (2017 AMC 10B, Problem #21) In $\triangle ABC$, $AB = 6$, $AC = 8$, $BC = 10$, and D is the midpoint of $\overline{BC}$. What is the sum of the radii of the circles inscribed in $\triangle ADB$ and $\triangle ADC$?

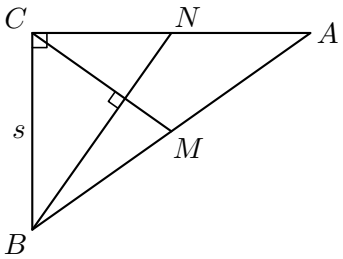
(A) $\sqrt{5}$ **(B)** $\frac{11}{4}$ **(C)** $2\sqrt{2}$ **(D)** $\frac{17}{6}$ **(E)** 3

Advanced.

11. (1986 AHSME, Problem #29) Two of the altitudes of the scalene triangle ABC have length 4 and 12. If the length of the third altitude is also an integer, what is the biggest it can be?

(A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** none of these

12. (1982 AHSME, Problem #21) In the adjoining figure, the triangle ABC is a right triangle with $\angle BCA = 90°$. Median CM is perpendicular to median BN, and side $BC = s$. The length of BN is



(A) $s\sqrt{2}$ **(B)** $\frac{3}{2}s\sqrt{2}$ **(C)** $2s\sqrt{2}$ **(D)** $\frac{s\sqrt{5}}{2}$ **(E)** $\frac{s\sqrt{6}}{2}$

13. (2018 AMC 10A, Problem #24) Triangle ABC with $AB = 50$ and $AC = 10$ has area 120. Let D be the midpoint of $\overline{AB}$, and let E be the midpoint of $\overline{AC}$. The angle bisector of $\angle BAC$ intersects $\overline{DE}$ and $\overline{BC}$ at F and G, respectively. What is the area of quadrilateral $FDBG$?

(A) 60 **(B)** 65 **(C)** 70 **(D)** 75 **(E)** 80

14. (2004 AMC 10B, Problem #22) A triangle with sides of 5, 12, and 13 has both an inscribed and a circumscribed circle. What is the distance between the centers of those circles?

(A) $\frac{3\sqrt{5}}{2}$ **(B)** $\frac{7}{2}$ **(C)** $\sqrt{15}$ **(D)** $\frac{\sqrt{65}}{2}$ **(E)** $\frac{9}{2}$

15. (1985 AIME, Problem #6) As shown in the figure, triangle ABC is divided into six smaller triangles by lines drawn from the vertices through a common interior point. The areas of four of these triangles are as indicated. Find the area of triangle ABC.

