TRIANGLE CENTERS, MEDIANS, AND CEVIANS FOR AMC 10: SOLUTIONS

ANDY JO

Contents

1. Solutions 2
2. Answer Key 10

1. Solutions

Beginner.

1. (2014 AMC 10A, Problem #9) The two legs of a right triangle, which are altitudes, have lengths $2\sqrt{3}$ and 6. How long is the third altitude of the triangle?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

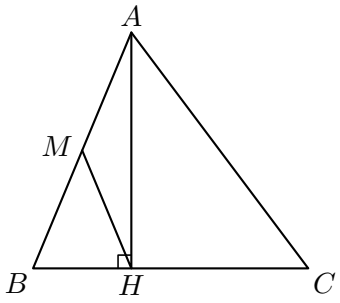
Strategy. The third altitude is the altitude to the hypotenuse. By Theorem 5.1.1 of the handout it equals $2K$ divided by the hypotenuse.

Solution. The hypotenuse has length $\sqrt{(2\sqrt{3})^2+6^2} = \sqrt{12+36} = 4\sqrt{3}$, and the area is $K = \frac{1}{2}\cdot 2\sqrt{3}\cdot 6 = 6\sqrt{3}$. By Theorem 5.1.1, the altitude to the hypotenuse is

$$\frac{2K}{4\sqrt{3}} = \frac{12\sqrt{3}}{4\sqrt{3}} = 3.$$

Answer: (C) 3. □

2. (1986 AHSME, Problem #11) In $\triangle ABC$, $AB = 13$, $BC = 14$ and $CA = 15$. Also, M is the midpoint of side AB and H is the foot of the altitude from A to BC. The length of HM is



(A) 6 **(B)** 6.5 **(C)** 7 **(D)** 7.5 **(E)** 8

Strategy. The segment HM is the median to the hypotenuse AB of the right triangle AHB.

Solution. Since $AH \perp BC$, triangle AHB has a right angle at H, and its hypotenuse is $AB = 13$. The point M is the midpoint of AB, so by Theorem 4.2.1 of the handout (the midpoint of the hypotenuse),

$$HM = \tfrac{1}{2}\,AB = 6.5.$$

The lengths $BC = 14$ and $CA = 15$ are not needed; they only fix the shape of the triangle.

Answer: (B) 6.5. □

3. (1964 AHSME, Problem #38) The sides PQ and PR of triangle PQR are respectively of lengths 4 inches and 7 inches. The median PM is $3\frac{1}{2}$ inches. Then QR, in inches, is:

(A) 6 **(B)** 7 **(C)** 8 **(D)** 9 **(E)** 10

Strategy. Use the median length formula, Theorem 2.2.1 of the handout, with $QM = \frac{1}{2}QR$.

Solution. By Theorem 2.2.1,

$$PQ^2 + PR^2 = 2PM^2 + 2QM^2, \qquad \text{so} \qquad 16 + 49 = 2\cdot\frac{49}{4} + 2QM^2.$$

Then $2QM^2 = 65 - \frac{49}{2} = \frac{81}{2}$, so $QM^2 = \frac{81}{4}$ and $QM = \frac{9}{2}$. Since M is the midpoint of QR, $QR = 2\,QM = 9$.

Answer: (D) 9. □

4. (1965 AHSME, Problem #16) Let line AC be perpendicular to line CE. Connect A to the midpoint D of CE, and connect E to the midpoint B of AC. If AD and EB intersect in point F, and $BC = CD = 15$ inches, then the area of triangle DFE, in square inches, is:

(A) 50 **(B)** $50\sqrt{2}$ **(C)** 75 **(D)** $\frac{15}{2}\sqrt{105}$ **(E)** 100

Strategy. Two medians of triangle ACE cross at F, so F is its centroid.

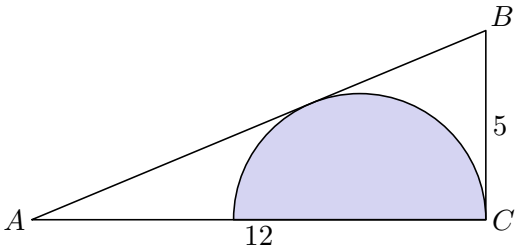
Solution. Since B is the midpoint of AC and D is the midpoint of CE, $AC = 2\,BC = 30$ and $CE = 2\,CD = 30$. The angle at C is right, so $[ACE] = \frac{1}{2} \cdot 30 \cdot 30 = 450$.

In triangle ACE, the segment AD joins A to the midpoint of CE, and EB joins E to the midpoint of AC. The point F where these two medians cross is the centroid. The triangle DFE has a vertex E of triangle ACE, the midpoint D of the side EC, and the centroid F, so it is one of the six triangles of equal area in Theorem 2.1.1 of the handout. Hence

$$[DFE] = \tfrac{1}{6}\,[ACE] = \tfrac{450}{6} = 75.$$

Answer: (C) 75. □

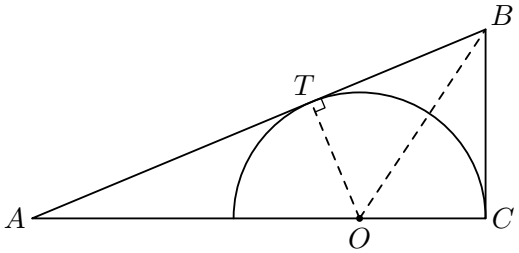
5. (2017 AMC 8, Problem #22) In the right triangle ABC, $AC = 12$, $BC = 5$, and angle C is a right angle. A semicircle is inscribed in the triangle as shown. What is the radius of the semicircle?



(A) $\frac{7}{6}$ **(B)** $\frac{13}{5}$ **(C)** $\frac{59}{18}$ **(D)** $\frac{10}{3}$ **(E)** $\frac{60}{13}$

Strategy. The center O lies on AC and is at distance r from both BC and BA. So BO bisects $\angle ABC$, and the angle bisector theorem gives CO.

Solution. By the Pythagorean theorem, $AB = \sqrt{12^2 + 5^2} = 13$. Let O be the center of the semicircle and r its radius. As the figure shows, the diameter lies on AC with one end at C, so O lies on AC with $OC = r$. The arc touches AB at a point T. A line that touches a circle is perpendicular to the radius at the point of contact, so $OT \perp AB$ and $OT = r$.



Since $OC \perp BC$, the point O is at distance $OC = r$ from line BC, and it is at distance $OT = r$ from line BA. The point O lies inside $\angle ABC$, so by Theorem 3.1.1 of the handout,

BO bisects $\angle ABC$. By the angle bisector theorem in triangle ABC,

$$CO : OA = BC : BA = 5 : 13, \qquad \text{so} \qquad r = CO = \frac{5}{18} \cdot 12 = \frac{10}{3}.$$

Areas give the same result: $[OBC] + [OBA] = [ABC]$, that is, $\frac{1}{2} \cdot 5r + \frac{1}{2} \cdot 13r = 30$, so $9r = 30$.

Answer: (D) $\dfrac{10}{3}$. □

Intermediate.

6. (2002 AMC 10B, Problem #22) Let $\triangle XOY$ be a right-angled triangle with $m\angle XOY = 90°$. Let M and N be the midpoints of legs OX and OY, respectively. Given that $XN = 19$ and $YM = 22$, what is XY?

(A) 24 **(B)** 26 **(C)** 28 **(D)** 30 **(E)** 32

Strategy. Each given median is the hypotenuse of a right triangle with one whole leg and one half leg, as in the remark after Theorem 2.2.1 of the handout. Add the two equations.

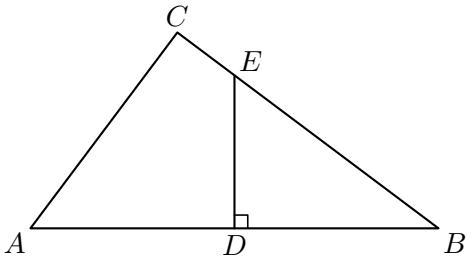
Solution. Let $OX = x$ and $OY = y$. The triangles XON and YOM have right angles at O, with $ON = \frac{y}{2}$ and $OM = \frac{x}{2}$, so

$$x^2 + \frac{y^2}{4} = 19^2 = 361 \quad \text{and} \quad y^2 + \frac{x^2}{4} = 22^2 = 484.$$

Adding, $\frac{5}{4}(x^2 + y^2) = 845$, so $x^2 + y^2 = 676$. The segment XY is the hypotenuse of triangle XOY, so $XY = \sqrt{676} = 26$. (Here $x = 16$ and $y = 2\sqrt{105}$.)

Answer: (B) 26. □

7. (1952 AHSME, Problem #24) In the figure, it is given that angle $C = 90°$, $AD = DB$, $DE \perp AB$, $AB = 20$, and $AC = 12$. The area of quadrilateral $ADEC$ is:



(A) 75 **(B)** $58\dfrac{1}{2}$ **(C)** 48 **(D)** $37\dfrac{1}{2}$ **(E)** None of these

Strategy. The segment DE lies on the perpendicular bisector of AB, so $EA = EB$. Find EB from the right triangle ACE.

Solution. By the Pythagorean theorem, $BC = \sqrt{20^2 - 12^2} = 16$, so $[ABC] = \frac{1}{2} \cdot 12 \cdot 16 = 96$. The point D is the midpoint of AB and $DE \perp AB$, so DE lies on the perpendicular bisector of AB, and $EA = EB$ by Theorem 4.1.1 of the handout. Let $EB = t$. Then $CE = 16 - t$, and the right triangle ACE gives

$$t^2 = EA^2 = 12^2 + (16 - t)^2 = 400 - 32t + t^2,$$

so $t = \frac{400}{32} = 12.5$. In the right triangle BDE, $BD = 10$ and $DE = \sqrt{12.5^2 - 10^2} = 7.5$, so $[BDE] = \frac{1}{2} \cdot 10 \cdot 7.5 = 37.5$. The quadrilateral $ADEC$ is triangle ABC without triangle BDE, so $[ADEC] = 96 - 37.5 = 58.5 = 58\frac{1}{2}$.

Answer: (B) $58\dfrac{1}{2}$. □

8. (2000 AMC 12, Problem #19) In triangle ABC, $AB = 13$, $BC = 14$, and $AC = 15$. Let D denote the midpoint of $\overline{BC}$ and let E denote the intersection of $\overline{BC}$ with the bisector of angle BAC. Which of the following is closest to the area of triangle ADE?

(A) 2 **(B)** 2.5 **(C)** 3 **(D)** 3.5 **(E)** 4

Strategy. The points D and E both lie on BC, so the height of triangle ADE is the altitude from A. Find BE with the angle bisector theorem and the altitude with the Pythagorean theorem.

Solution. Let H be the foot of the altitude from A, and let $x = BH$. The right triangles ABH and ACH share the leg AH, so

$$13^2 - x^2 = 15^2 - (14 - x)^2,$$

which gives $169 = 29 + 28x$, so $x = 5$ and $AH = \sqrt{13^2 - 5^2} = 12$. By the angle bisector theorem,

$$BE = \frac{BC \cdot AB}{AB + AC} = \frac{14 \cdot 13}{28} = 6.5,$$

and $BD = 7$, so $DE = 0.5$. Triangle ADE has its base DE on line BC and height $AH = 12$, so

$$[ADE] = \tfrac{1}{2} \cdot 0.5 \cdot 12 = 3.$$

The area is exactly 3.

Answer: (C) 3. □

9. (1973 AHSME, Problem #9) In $\triangle ABC$ with right angle at C, altitude CH and median CM trisect the right angle. If the area of $\triangle CHM$ is K, then the area of $\triangle ABC$ is

(A) $6K$ **(B)** $4\sqrt{3}K$ **(C)** $3\sqrt{3}K$ **(D)** $3K$ **(E)** $4K$

Strategy. By Theorem 4.2.1 of the handout, $CM = \frac{1}{2}AB$. The right triangle CHM has a 30° angle at C, which gives HM in terms of CM. Triangles CHM and CAB have the same height CH, so Theorem 1.1.1 relates their areas to the bases HM and AB.

Solution. The altitude CH and the median CM trisect the right angle at C, so each of the three angles at C is 30°. The median CM goes to the midpoint of the hypotenuse, so $CM = MA = MB = \frac{1}{2}AB$ by Theorem 4.2.1. In the right triangle CHM, the angle at C is 30°, and the leg opposite a 30° angle is half the hypotenuse, so

$$HM = \tfrac{1}{2}\,CM = \tfrac{1}{4}\,AB.$$

Triangles CHM and CAB have the same height CH to line AB, so by Theorem 1.1.1,

$$[ABC] = \frac{AB}{HM} \cdot [CHM] = 4K.$$

Answer: (E) $4K$. □

10. (2017 AMC 10B, Problem #21) In $\triangle ABC$, $AB = 6$, $AC = 8$, $BC = 10$, and D is the midpoint of $\overline{BC}$. What is the sum of the radii of the circles inscribed in $\triangle ADB$ and $\triangle ADC$?

(A) $\sqrt{5}$ **(B)** $\frac{11}{4}$ **(C)** $2\sqrt{2}$ **(D)** $\frac{17}{6}$ **(E)** 3

Strategy. The angle at A is right, so $AD = 5$. The median AD halves the area. Then $r = \frac{K}{s}$ for each of the two triangles.

Solution. Since $6^2 + 8^2 = 10^2$, the angle at A is right and BC is the hypotenuse. By Theorem 4.2.1 of the handout, $AD = \frac{1}{2}\,BC = 5$. The area of triangle ABC is $\frac{1}{2} \cdot 6 \cdot 8 = 24$, and the median AD divides it into two triangles of area 12 (Corollary 1.1.2).

Triangle ADB has sides 5, 5 and 6, so its semiperimeter is 8, and by Theorem 3.2.2 its inradius is $\frac{12}{8} = \frac{3}{2}$. Triangle ADC has sides 5, 5 and 8, semiperimeter 9 and inradius $\frac{12}{9} = \frac{4}{3}$. The sum of the radii is

$$\frac{3}{2} + \frac{4}{3} = \frac{17}{6}.$$

Answer: (D) $\dfrac{17}{6}$. □

Advanced.

11. (1986 AHSME, Problem #29) Two of the altitudes of the scalene triangle ABC have length 4 and 12. If the length of the third altitude is also an integer, what is the biggest it can be?

(A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** none of these

Strategy. By Theorem 5.1.1 of the handout, the sides are proportional to the reciprocals of the altitudes. Apply the triangle inequality.

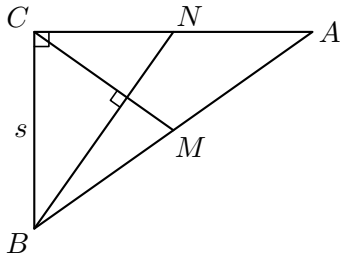
Solution. Let the third altitude be h and the area K. By Theorem 5.1.1 the sides are $\frac{2K}{4}$, $\frac{2K}{12}$ and $\frac{2K}{h}$, which are proportional to 3, 1 and $\frac{12}{h}$. The triangle inequality requires

$$3 - 1 < \frac{12}{h} < 3 + 1,$$

so $3 < h < 6$, and the integer h is 4 or 5. If $h = 4$, two altitudes would be equal, so two sides would be equal; but the triangle is scalene. Hence $h = 5$, with sides in the ratio $3 : 1 : \frac{12}{5}$, that is, $15 : 5 : 12$, and these lengths do form a scalene triangle.

Answer: (B) 5. □

12. (1982 AHSME, Problem #21) In the adjoining figure, the triangle ABC is a right triangle with $\angle BCA = 90°$. Median CM is perpendicular to median BN, and side $BC = s$. The length of BN is



(A) $s\sqrt{2}$ **(B)** $\dfrac{3}{2}s\sqrt{2}$ **(C)** $2s\sqrt{2}$ **(D)** $\dfrac{s\sqrt{5}}{2}$ **(E)** $\dfrac{s\sqrt{6}}{2}$

Strategy. Let G be the centroid. Then $CG = \frac{2}{3}\,CM$ and $BG = \frac{2}{3}\,BN$, and both medians have lengths known in terms of the legs. The right angle at G gives an equation.

Solution. Let $AC = b$, and let G be the point where the medians CM and BN meet, the centroid of triangle ABC. By Theorem 4.2.1 of the handout, $CM = \frac{1}{2}\,AB$, so by Theorem 2.1.1

$$CG = \tfrac{2}{3}\,CM = \tfrac{1}{3}\,AB \quad \text{and} \quad CG^2 = \frac{AB^2}{9} = \frac{s^2 + b^2}{9}.$$

The median BN is the hypotenuse of the right triangle BCN with legs s and $\frac{b}{2}$, so $BN^2 = s^2 + \frac{b^2}{4}$, and

$$BG^2 = \tfrac{4}{9}\, BN^2 = \frac{4s^2 + b^2}{9}.$$

The medians are perpendicular, so triangle BGC has a right angle at G, and $BG^2 + CG^2 = BC^2$:

$$\frac{4s^2 + b^2}{9} + \frac{s^2 + b^2}{9} = s^2, \qquad \text{so} \qquad 5s^2 + 2b^2 = 9s^2,$$

and $b^2 = 2s^2$. Then $BN^2 = s^2 + \frac{s^2}{2} = \frac{3}{2}\, s^2$, and $BN = \frac{s\sqrt{6}}{2}$.

Answer: (E) $\dfrac{s\sqrt{6}}{2}$. □

13. (2018 AMC 10A, Problem #24) Triangle ABC with $AB = 50$ and $AC = 10$ has area 120. Let D be the midpoint of $\overline{AB}$, and let E be the midpoint of $\overline{AC}$. The angle bisector of $\angle BAC$ intersects $\overline{DE}$ and $\overline{BC}$ at F and G, respectively. What is the area of quadrilateral $FDBG$?

(A) 60 **(B)** 65 **(C)** 70 **(D)** 75 **(E)** 80

Strategy. The quadrilateral $FDBG$ is triangle ABG with triangle ADF removed. Find both areas with the angle bisector theorem, in triangles ABC and ADE, and Theorem 1.1.1 of the handout.

Solution. By the angle bisector theorem in triangle ABC, $BG : GC = AB : AC = 50 : 10 = 5 : 1$. Triangles ABG and ABC share the vertex A and have their bases on line BC, so by Theorem 1.1.1,

$$[ABG] = \tfrac{5}{6} \cdot 120 = 100.$$

Next, D is the midpoint of AB, so $[ADC] = \frac{1}{2}[ABC] = 60$ (triangles ADC and ABC share the vertex C). In the same way, E is the midpoint of AC, so $[ADE] = \frac{1}{2}[ADC] = 30$ (triangles ADE and ADC share the vertex D). In triangle ADE, the segment AF bisects $\angle DAE$, which is $\angle BAC$, so by the angle bisector theorem $DF : FE = AD : AE = 25 : 5 = 5 : 1$, and

$$[ADF] = \tfrac{5}{6} \cdot 30 = 25.$$

The point D lies on AB and F lies on AG, so triangle ADF lies inside triangle ABG, and the rest of triangle ABG is the quadrilateral $FDBG$. Hence

$$[FDBG] = [ABG] - [ADF] = 100 - 25 = 75.$$

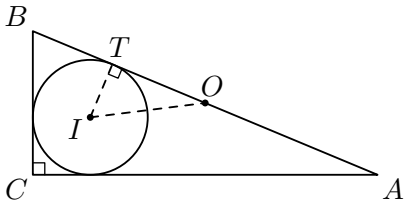
Answer: (D) 75. □

14. (2004 AMC 10B, Problem #22) A triangle with sides of 5, 12, and 13 has both an inscribed and a circumscribed circle. What is the distance between the centers of those circles?

(A) $\dfrac{3\sqrt{5}}{2}$ **(B)** $\dfrac{7}{2}$ **(C)** $\sqrt{15}$ **(D)** $\dfrac{\sqrt{65}}{2}$ **(E)** $\dfrac{9}{2}$

Strategy. The side 13 is the hypotenuse. By Theorem 4.2.1 of the handout its midpoint is the circumcenter O, and the tangent lengths locate the point T where the incircle touches it. The incenter I is at distance r from T on the perpendicular to the hypotenuse, so the right triangle ITO gives IO.

Solution. The sides satisfy $5^2 + 12^2 = 169 = 13^2$, so by the converse of the Pythagorean theorem the angle opposite the side 13 is right. Name the triangle ABC with the right angle at C, $BC = 5$, $CA = 12$ and $AB = 13$; then $s = 15$. By Theorem 4.2.1, the circumcenter O is the midpoint of AB, so $AO = 6.5$. By Corollary 3.2.4, the inradius is $r = \frac{5+12-13}{2} = 2$. The incircle touches AB at a point T with $AT = s - a = 15 - 5 = 10$ by Theorem 3.2.3, and $IT \perp AB$ with $IT = r = 2$.

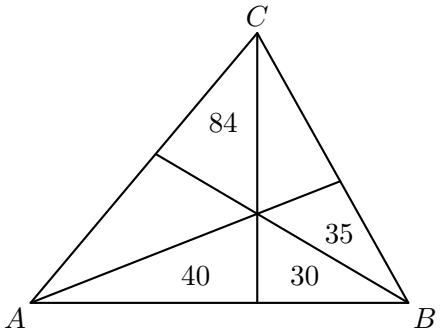


Both O and T lie on AB, with $AO = 6.5$ and $AT = 10$, so $OT = 3.5$. The right triangle ITO gives

$$IO = \sqrt{IT^2 + OT^2} = \sqrt{4 + \tfrac{49}{4}} = \sqrt{\tfrac{65}{4}} = \frac{\sqrt{65}}{2}.$$

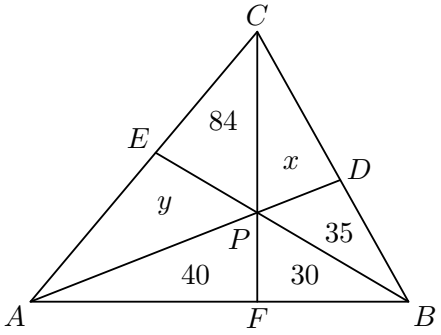
Answer: (D) $\dfrac{\sqrt{65}}{2}$. □

15. (1985 AIME, Problem #6) As shown in the figure, triangle ABC is divided into six smaller triangles by lines drawn from the vertices through a common interior point. The areas of four of these triangles are as indicated. Find the area of triangle ABC.



Strategy. Let x and y be the two unknown areas. Theorem 1.2.1 of the handout on two of the cevians, compared with Theorem 1.1.1 for the small triangles along the same side, gives two equations.

Solution. Let the lines from A, B and C meet the opposite sides at D, E and F, and let them meet at P, as in the figure below. Then $[APF] = 40$, $[FPB] = 30$, $[BPD] = 35$ and $[CPE] = 84$. Let $x = [DPC]$ and $y = [EPA]$.



The point P lies on the cevian AD, so by Theorem 1.2.1,

$$BD : DC = [ABP] : [ACP] = 70 : (y + 84).$$

Triangles PBD and PDC have the common vertex P, so by Theorem 1.1.1, $BD : DC = 35 : x$. Hence $70x = 35(y + 84)$, that is, $y = 2x - 84$.

The point P also lies on the cevian BE, so

$$CE : EA = [BCP] : [BAP] = (35 + x) : 70,$$

and triangles PCE and PEA give $CE : EA = 84 : y$. Hence $y(35 + x) = 84 \cdot 70 = 5880$. Substituting $y = 2x - 84$,

$$(2x - 84)(x + 35) = 5880, \qquad 2x^2 - 14x - 2940 = 5880, \qquad x^2 - 7x - 4410 = 0.$$

This factors as $(x - 70)(x + 63) = 0$, and $x > 0$, so $x = 70$ and $y = 56$. The area of triangle ABC is

$$40 + 30 + 35 + 70 + 84 + 56 = 315.$$

As a check, Theorem 1.1.1 on the third cevian CF gives $AF : FB = [APF] : [FPB] = 40 : 30 = 4 : 3$, and Theorem 1.2.1 gives the same ratio, $AF : FB = [CAP] : [CBP] = 140 : 105 = 4 : 3$.

Answer: 315. □

2. Answer Key

Problem	Letter	Value
1	C	3
2	B	6.5
3	D	9
4	C	75
5	D	$\frac{10}{3}$
6	B	26
7	B	$58\frac{1}{2}$
8	C	3
9	E	$4K$
10	D	$\frac{17}{6}$
11	B	5
12	E	$\frac{s\sqrt{6}}{2}$
13	D	75
14	D	$\frac{\sqrt{65}}{2}$
15		315