TRIANGLE CENTERS, MEDIANS, AND CEVIANS FOR AMC 10: PRACTICE SET SOLUTIONS

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1. Solutions

Beginner.

1. (1966 AHSME, Problem #14) The length of rectangle $ABCD$ is 5 inches and its width is 3 inches. Diagonal AC is divided into three equal segments by points E and F. The area of triangle BEF, expressed in square inches, is:

(A) $\frac{3}{2}$ **(B)** $\frac{5}{3}$ **(C)** $\frac{5}{2}$ **(D)** $\frac{1}{3}\sqrt{34}$ **(E)** $\frac{1}{3}\sqrt{68}$

Strategy. Triangles BEF and BAC have the common vertex B, and their bases EF and AC lie on one line. The equal-heights theorem compares their areas.

Solution. The diagonal AC cuts the rectangle into two congruent right triangles, so

$$[BAC] = \tfrac{1}{2} \cdot AB \cdot BC = \tfrac{1}{2} \cdot 5 \cdot 3 = \tfrac{15}{2}.$$

The points E and F lie on AC, so triangles BEF and BAC have the same height from B to line AC. Since $EF = \frac{1}{3}AC$, the equal-heights theorem gives

$$[BEF] = \frac{EF}{AC} \cdot [BAC] = \frac{1}{3} \cdot \frac{15}{2} = \frac{5}{2}.$$

Answer: (C) $\frac{5}{2}$. □

2. (1966 AHSME, Problem #11) The sides of triangle BAC are in the ratio $2 : 3 : 4$. BD is the angle-bisector drawn to the shortest side AC, dividing it into segments AD and CD. If the length of AC is 10, then the length of the longer segment of AC is:

(A) $3\frac{1}{2}$ **(B)** 5 **(C)** $5\frac{5}{7}$ **(D)** 6 **(E)** $7\frac{1}{2}$

Strategy. The shortest side fixes the scale, so the other two sides are known up to their order. The angle bisector theorem divides AC in the ratio of the two sides at B.

Solution. The shortest side $AC = 10$ corresponds to the 2 in the ratio $2 : 3 : 4$, so the sides of the triangle are 10, 15 and 20, and BA and BC are 15 and 20 in some order. The segment BD bisects $\angle ABC$, so the angle bisector theorem gives

$$AD : DC = BA : BC,$$

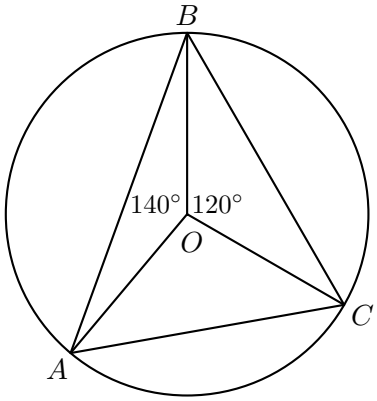
which is $3 : 4$ or $4 : 3$. In both cases $AC = 10$ is divided into the pieces

$$\frac{3}{7} \cdot 10 = \frac{30}{7} \quad \text{and} \quad \frac{4}{7} \cdot 10 = \frac{40}{7}.$$

The longer piece is $\frac{40}{7} = 5\frac{5}{7}$.

Answer: (C) $5\frac{5}{7}$. □

3. (2007 AMC 10B, Problem #4) The point O is the center of the circle circumscribed about $\triangle ABC$, with $\angle BOC = 120^\circ$ and $\angle AOB = 140^\circ$, as shown. What is the degree measure of $\angle ABC$?



(A) 35 **(B)** 40 **(C)** 45 **(D)** 50 **(E)** 60

Strategy. The circumcenter is equally far from the three vertices, so triangles OAB and OBC are isosceles. Their base angles at B together make up $\angle ABC$.

Solution. By the circumcenter theorem, $OA = OB = OC$. Triangle OAB is isosceles with vertex angle $\angle AOB = 140^\circ$, so its base angles are

$$\angle OBA = \angle OAB = \frac{180^\circ - 140^\circ}{2} = 20^\circ.$$

Triangle OBC is isosceles with vertex angle $\angle BOC = 120^\circ$, so

$$\angle OBC = \angle OCB = \frac{180^\circ - 120^\circ}{2} = 30^\circ.$$

As the figure shows, O lies inside triangle ABC, so the ray BO splits $\angle ABC$ into these two angles:

$$\angle ABC = \angle ABO + \angle OBC = 20^\circ + 30^\circ = 50^\circ.$$

Answer: (D) 50. □

4. (1964 AHSME, Problem #13) A circle is inscribed in a triangle with side lengths 8, 13, and 17. Let the segments of the side of length 8, made by a point of tangency, be r and s, with $r < s$. What is the ratio $r : s$?

(A) 1 : 3 **(B)** 2 : 5 **(C)** 1 : 2 **(D)** 2 : 3 **(E)** 3 : 4

Strategy. Each of the two pieces is the tangent length from one end of the side, and the tangent lengths theorem gives it from the semiperimeter.

Solution. Name the triangle ABC with $BC = 8$, $CA = 13$ and $AB = 17$, and let the inscribed circle touch BC at D. The semiperimeter is $\frac{1}{2}(8 + 13 + 17) = 19$, so by the tangent lengths theorem

$$BD = 19 - CA = 6 \quad \text{and} \quad CD = 19 - AB = 2.$$

Exchanging the names of the sides 13 and 17 only exchanges BD and CD. Hence $r = 2$ and $s = 6$, and $r : s = 1 : 3$.

Answer: (A) 1 : 3. □

5. (1961 AHSME, Problem #21) Medians AD and CE of triangle ABC intersect in M. The midpoint of AE is N. Let the area of triangle MNE be k times the area of triangle ABC. Then k equals:

(A) $\frac{1}{6}$ **(B)** $\frac{1}{8}$ **(C)** $\frac{1}{9}$ **(D)** $\frac{1}{12}$ **(E)** $\frac{1}{16}$

Strategy. Two medians meet at the centroid, so M is the centroid. Triangle AME is one of the six equal triangles cut by the medians, and MN is a median of it.

Solution. Let $K = [ABC]$. The medians AD and CE meet at the centroid, so M is the centroid of triangle ABC. Since CE is a median, E is the midpoint of AB. So the vertices of triangle AME are the vertex A, the midpoint E of the side AB and the centroid M, and by the centroid theorem triangle AME is one of the six triangles of area $\frac{K}{6}$ cut by the three medians.

The point N is the midpoint of AE, so MN is a median of triangle AME, and a median divides a triangle into two triangles of equal area. Hence

$$[MNE] = \tfrac{1}{2}\,[AME] = \tfrac{1}{2} \cdot \tfrac{K}{6} = \tfrac{K}{12},$$

and $k = \frac{1}{12}$.

Answer: (D) $\frac{1}{12}$. □

Intermediate.

6. (2024 AMC 10B, Problem #10) Quadrilateral $ABCD$ is a parallelogram, and E is the midpoint of the side $\overline{AD}$. Let F be the intersection of lines EB and AC. What is the ratio of the area of quadrilateral $CDEF$ to the area of triangle CFB?

(A) 5 : 4 **(B)** 4 : 3 **(C)** 3 : 2 **(D)** 5 : 3 **(E)** 2 : 1

Strategy. The diagonals of a parallelogram bisect each other, so line AC contains a median of triangle ABD, and F is the centroid of that triangle. Then every area in the problem is a simple fraction of the area of the parallelogram.

Solution. Let the diagonals AC and BD meet at O. The diagonals of a parallelogram bisect each other, so O is the midpoint of both BD and AC. In triangle ABD, the segment BE is a median because E is the midpoint of AD, and the segment AO is a median because O is the midpoint of BD. These two medians meet at F, so F is the centroid of triangle ABD, and by the centroid theorem

$$AF = \tfrac{2}{3}\,AO = \tfrac{1}{3}\,AC, \qquad \text{so} \qquad CF = \tfrac{2}{3}\,CA.$$

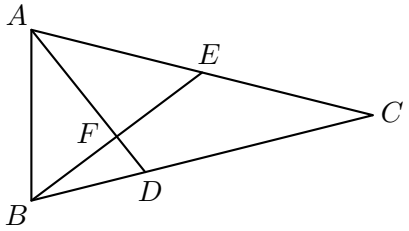
Let the parallelogram have area 12. Each diagonal divides it into two congruent triangles, so $[ABD] = [ABC] = [ACD] = 6$.

Triangles CFB and CAB have the common vertex B and their bases CF and CA on line AC, so by the equal-heights theorem $[CFB] = \frac{2}{3} \cdot 6 = 4$.

Triangle AEF has as its vertices the vertex A of triangle ABD, the midpoint E of its side AD and its centroid F, so by the centroid theorem $[AEF] = \frac{1}{6}[ABD] = 1$. The quadrilateral $CDEF$ is triangle ACD with triangle AEF removed, so $[CDEF] = 6-1 = 5$, and the ratio is 5 : 4.

Answer: (A) 5 : 4. □

7. (1980 AHSME, Problem #21) In triangle ABC, $\angle CBA = 72°$, E is the midpoint of side AC, and D is a point on side BC such that $2BD = DC$; AD and BE intersect at F. The ratio of the area of $\triangle BDF$ to the area of quadrilateral $FDCE$ is



(A) $\frac{1}{5}$ **(B)** $\frac{1}{4}$ **(C)** $\frac{1}{3}$ **(D)** $\frac{2}{5}$ **(E)** none of these

Strategy. The point F lies on two cevians. The point-on-a-cevian theorem for each of them gives the ratios of $[ABF]$, $[CBF]$ and $[ACF]$, and the equal-heights theorem splits these triangles further.

Solution. Let $x = [ABF]$. The point F lies on the cevian BE, so by the point-on-a-cevian theorem

$$[ABF] : [CBF] = AE : EC = 1 : 1,$$

and $[CBF] = x$. The point F also lies on the cevian AD, so

$$[ABF] : [ACF] = BD : DC = 1 : 2,$$

and $[ACF] = 2x$.

Triangles BDF and CDF have the common vertex F and their bases on line BC, with $BD : DC = 1 : 2$, and together they make up triangle CBF. By the equal-heights theorem, $[BDF] = \frac{x}{3}$ and $[CDF] = \frac{2x}{3}$. Since E is the midpoint of AC, the segment FE is a median of triangle ACF, so $[CEF] = \frac{1}{2}[ACF] = x$. The diagonal FC divides the quadrilateral $FDCE$ into triangles CDF and CEF, so

$$[FDCE] = \frac{2x}{3} + x = \frac{5x}{3} \qquad \text{and} \qquad \frac{[BDF]}{[FDCE]} = \frac{x/3}{5x/3} = \frac{1}{5}.$$

The angle of 72° is not needed.

Answer: (A) $\dfrac{1}{5}$. □

8. (2018 AMC 10A, Problem #16) Right triangle ABC has leg lengths $AB = 20$ and $BC = 21$. Including $\overline{AB}$ and $\overline{BC}$, how many line segments with integer length can be drawn from vertex B to a point on hypotenuse $\overline{AC}$?

(A) 5 **(B)** 8 **(C)** 12 **(D)** 13 **(E)** 15

Strategy. The shortest segment from B to AC is the altitude, whose length is twice the area divided by AC. On each side of the foot of the altitude, the length of the segment increases steadily up to a leg.

Solution. The hypotenuse is $AC = \sqrt{20^2 + 21^2} = \sqrt{841} = 29$, and $[ABC] = \frac{1}{2} \cdot 20 \cdot 21 = 210$. By the altitudes-and-sides theorem, the altitude from B has length

$$h = \frac{2 \cdot 210}{29} = \frac{420}{29},$$

and $14 < h < 15$ because $14 \cdot 29 = 406$ and $15 \cdot 29 = 435$. The angles at A and C are acute, so the foot H of this altitude lies on AC, between A and C.

For a point X of AC, the right triangle BHX gives $BX^2 = h^2 + HX^2$. As X moves from H to A, the distance HX increases from 0 to HA, so BX increases from h to $BA = 20$ and takes each value in between exactly once. The integers in this range are $15, 16, \ldots, 20$, which gives 6 segments. In the same way, as X moves from H to C, the length BX increases from h to $BC = 21$, which gives the 7 integer lengths $15, 16, \ldots, 21$. The point H itself gives the length h, which is not an integer. The total is $6 + 7 = 13$.

Answer: (D) 13. □

9. (2003 AMC 10A, Problem #17) The number of inches in the perimeter of an equilateral triangle equals the number of square inches in the area of its circumscribed circle. What is the radius, in inches, of the circle?

(A) $\frac{3\sqrt{2}}{\pi}$ **(B)** $\frac{3\sqrt{3}}{\pi}$ **(C)** $\sqrt{3}$ **(D)** $\frac{6}{\pi}$ **(E)** $\sqrt{3}\pi$

Strategy. Each vertex of an equilateral triangle is equally far from the other two, so the perpendicular bisector of each side contains the median from the opposite vertex. Then the circumcenter is the centroid, and the circumradius is two thirds of a median.

Solution. Let the triangle be ABC with side length x, let M be the midpoint of BC, and let O and R be the center and the radius of the circumscribed circle. Since $AB = AC$, the perpendicular bisector theorem puts A on the perpendicular bisector of BC. That line also passes through M, so it is the line AM, and the median AM is perpendicular to BC. In the same way, the perpendicular bisectors of CA and AB contain the medians from B and C. By the circumcenter theorem, O lies on all three perpendicular bisectors, hence on all three medians, so O is the centroid of triangle ABC.

The right triangle AMB gives

$$AM = \sqrt{x^2 - \left(\frac{x}{2}\right)^2} = \frac{\sqrt{3}}{2}\, x.$$

By the centroid theorem, $AO = \frac{2}{3}\, AM$, so

$$R = AO = \frac{2}{3} \cdot \frac{\sqrt{3}}{2}\, x = \frac{x}{\sqrt{3}}.$$

The perimeter $3x$ equals the area $\pi R^2 = \frac{\pi x^2}{3}$ of the circle, so

$$3x = \frac{\pi x^2}{3}, \qquad x = \frac{9}{\pi}, \qquad \text{and} \qquad R = \frac{x}{\sqrt{3}} = \frac{9}{\sqrt{3}\,\pi} = \frac{3\sqrt{3}}{\pi}.$$

Answer: (B) $\frac{3\sqrt{3}}{\pi}$. □

10. (2024 AMC 12B, Problem #20) Suppose A, B, and C are points in the plane with $AB = 40$ and $AC = 42$, and let x be the length of the line segment from A to the midpoint of $\overline{BC}$. Define a function f by letting $f(x)$ be the area of $\triangle ABC$. Then the domain of f is an open interval (p, q), and the maximum value r of $f(x)$ occurs at $x = s$. What is $p + q + r + s$?

(A) 909 **(B)** 910 **(C)** 911 **(D)** 912 **(E)** 913

Strategy. The median length formula turns the possible lengths of BC into the possible values of x. The area is largest when the angle at A is right, and then the median from A is half of the hypotenuse.

Solution. Let $a = BC$ and let M be the midpoint of BC, so that $x = AM$. By the median length formula,

$$x^2 = \frac{2 \cdot 40^2 + 2 \cdot 42^2 - a^2}{4} = \frac{6728 - a^2}{4}.$$

By the triangle inequality, the three points form a triangle exactly when $42 - 40 < a < 42 + 40$, that is, $2 < a < 82$, and every such a occurs. At the two ends of this interval,

$$\frac{6728 - 2^2}{4} = 1681 = 41^2 \qquad \text{and} \qquad \frac{6728 - 82^2}{4} = 1 = 1^2,$$

so the possible values of x form the open interval $(1, 41)$. Each x determines a, hence the three sides and the area, so f is well defined, and $p = 1$ and $q = 41$.

The area is $\frac{1}{2} \cdot AB \cdot h$, where h is the distance from C to line AB. The perpendicular from C to line AB is the shortest segment from C to that line, so $h \leq AC = 42$, with equality exactly when $AC \perp AB$. So the maximum area is

$$r = \tfrac{1}{2} \cdot 40 \cdot 42 = 840,$$

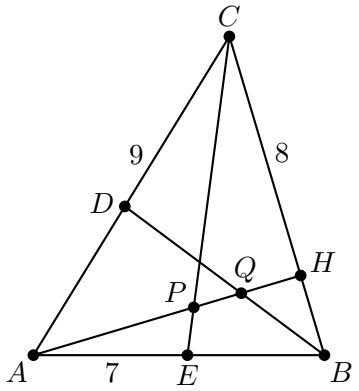
reached when $\angle BAC = 90°$. Then BC is the hypotenuse of the right triangle ABC, with $BC = \sqrt{40^2 + 42^2} = \sqrt{3364} = 58$, and by the midpoint-of-the-hypotenuse theorem the median AM is half of it, so $s = 29$. The median length formula agrees, since $6728 - 58^2 = 3364 = 4 \cdot 29^2$.

Therefore $p + q + r + s = 1 + 41 + 840 + 29 = 911$.

Answer: (C) 911. □

Advanced.

11. (2016 AMC 12B, Problem #17) In $\triangle ABC$ shown in the figure, $AB = 7$, $BC = 8$, $CA = 9$, and $\overline{AH}$ is an altitude. Points D and E lie on sides $\overline{AC}$ and $\overline{AB}$, respectively, so that $\overline{BD}$ and $\overline{CE}$ are angle bisectors, intersecting $\overline{AH}$ at Q and P, respectively. What is PQ?



(A) 1 **(B)** $\frac{5}{8}\sqrt{3}$ **(C)** $\frac{4}{5}\sqrt{2}$ **(D)** $\frac{8}{15}\sqrt{5}$ **(E)** $\frac{6}{5}$

Strategy. Find BH, HC and AH with the Pythagorean theorem. Then BQ bisects the angle at B of the right triangle ABH, and CP bisects the angle at C of the right triangle ACH, so the angle bisector theorem places Q and P on AH.

Solution. Measure positions on line BC from B toward C, so that C is at 8 and H is at some number t. Then $BH^2 = t^2$ and $CH^2 = (8 - t)^2$ wherever H lies on the line, and the right triangles AHB and AHC give

$$AH^2 = 7^2 - t^2 = 9^2 - (8 - t)^2.$$

This simplifies to $49 = 17 + 16t$, so $t = 2$. Hence H lies between B and C, with $BH = 2$, $HC = 6$ and

$$AH = \sqrt{49 - 4} = 3\sqrt{5}.$$

Since H lies on side BC, $\angle ABH = \angle ABC$. So BQ bisects the angle B of triangle ABH and meets the opposite side AH at Q, and the angle bisector theorem in triangle ABH gives

$$AQ : QH = BA : BH = 7 : 2, \qquad \text{so} \qquad QH = \frac{2}{9} \cdot 3\sqrt{5} = \frac{2\sqrt{5}}{3}.$$

In the same way, CP bisects the angle C of triangle ACH, and

$$AP : PH = CA : CH = 9 : 6 = 3 : 2, \qquad \text{so} \qquad PH = \frac{2}{5} \cdot 3\sqrt{5} = \frac{6\sqrt{5}}{5}.$$

Both points lie on segment AH, and $PH > QH$, so

$$PQ = PH - QH = \frac{6\sqrt{5}}{5} - \frac{2\sqrt{5}}{3} = \frac{18\sqrt{5} - 10\sqrt{5}}{15} = \frac{8}{15}\sqrt{5}.$$

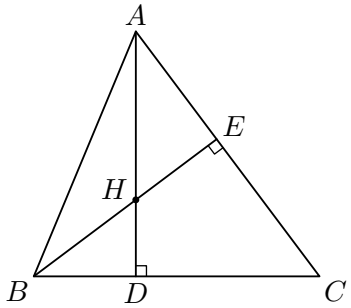
Answer: (D) $\frac{8}{15}\sqrt{5}$. □

12. (1964 AHSME, Problem #35) The sides of a triangle are of lengths 13, 14, and 15. The altitudes of the triangle meet at point H. If AD is the altitude to the side of length 14, what is the ratio $HD : HA$?

(A) 3 : 11 **(B)** 5 : 11 **(C)** 1 : 2 **(D)** 2 : 3 **(E)** 25 : 33

Strategy. Find the feet of two altitudes with the Pythagorean theorem. The orthocenter H lies on both altitudes, so the point-on-a-cevian theorem gives the ratios of the areas of triangles HAB, HAC and HBC, and $HD : AD = [HBC] : [ABC]$.

Solution. Let $BC = 14$, and name the other two vertices so that $AB = 13$ and $CA = 15$; the other choice gives the mirror image and the same ratio. Let E be the foot of the altitude from B, as in the figure.



Measure positions on line BC from B toward C, with D at t. The right triangles ADB and ADC give $AD^2 = 13^2 - t^2 = 15^2 - (14 - t)^2$, so $169 = 29 + 28t$ and $t = 5$. Hence $BD = 5$, $DC = 9$, $AD = \sqrt{169 - 25} = 12$, and the area is $K = \frac{1}{2} \cdot 14 \cdot 12 = 84$.

By the altitudes-and-sides theorem, the altitude from B has length $BE = \frac{2K}{CA} = \frac{168}{15} = \frac{56}{5}$, and the right triangles BEA and BEC give

$$AE = \sqrt{13^2 - \left(\frac{56}{5}\right)^2} = \sqrt{\frac{1089}{25}} = \frac{33}{5}, \qquad CE = \sqrt{14^2 - \left(\frac{56}{5}\right)^2} = \sqrt{\frac{1764}{25}} = \frac{42}{5}.$$

Since $AE + EC = 15 = AC$, the point E lies on side AC. By the orthocenter theorem, H is the point where the cevians AD and BE cross. Since H lies on both cevians, the

point-on-a-cevian theorem gives

$$[HAB] : [HAC] = BD : DC = 5 : 9,$$
$$[HBA] : [HBC] = AE : EC = 33 : 42 = 11 : 14.$$

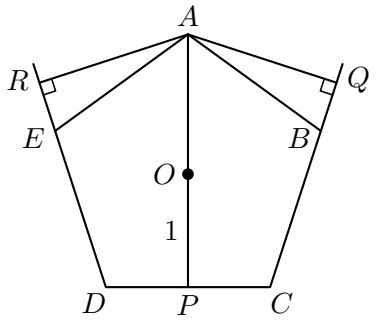
Combining the two ratios, $[HAB] : [HAC] : [HBC] = 55 : 99 : 70$, and these three triangles make up triangle ABC. Triangles HBC and ABC have the common base BC and the heights HD and AD, so

$$\frac{HD}{AD} = \frac{[HBC]}{[ABC]} = \frac{70}{55+99+70} = \frac{70}{224} = \frac{5}{16}.$$

Hence $HD = \frac{5}{16}AD$ and $HA = \frac{11}{16}AD$, so $HD : HA = 5 : 11$. In lengths, $HD = \frac{15}{4}$ and $HA = \frac{33}{4}$.

Answer: (B) $5 : 11$. □

13. (1986 AHSME, Problem #28) $ABCDE$ is a regular pentagon. AP, AQ and AR are the perpendiculars dropped from A onto CD, CB extended and DE extended, respectively. Let O be the center of the pentagon. If $OP = 1$, then $AO + AQ + AR$ equals



(A) 3 **(B)** $1+\sqrt{5}$ **(C)** 4 **(D)** $2+\sqrt{5}$ **(E)** 5

Strategy. Compute the area of the pentagon in two ways, as in the proof of $K = rs$: from the center O, which is at distance 1 from every side, and from the vertex A, whose two diagonals cut the pentagon into three triangles with heights AQ, AP and AR.

Solution. Let x be the side length. Triangles ABC and AED are congruent (SAS: $AB = AE$, $BC = ED$, and the angles at B and E are both $108°$), so $AC = AD$. The center O is also equally far from C and D. By the perpendicular bisector theorem, both A and O lie on the perpendicular bisector of CD. This line passes through A and is perpendicular to CD, so it meets CD at the foot P of the perpendicular from A. Hence P is the midpoint of CD, the center O, which is inside the pentagon, lies on segment AP, and O is at distance $OP = 1$ from the side CD.

The segments from O to the five vertices cut the pentagon into five congruent isosceles triangles, so O is at distance 1 from every side, and

$$[ABCDE] = 5 \cdot \tfrac{1}{2} \cdot x \cdot 1 = \tfrac{5}{2}\, x.$$

The two diagonals from A split $ABCDE$ into the three triangles ABC, ACD and ADE. Each has a side of the pentagon as its base, and its height from A is AQ (to line CB), AP (to CD) or AR (to line DE). So

$$[ABCDE] = \tfrac{1}{2}\, x \cdot AQ + \tfrac{1}{2}\, x \cdot AP + \tfrac{1}{2}\, x \cdot AR = \tfrac{x}{2}\,(AQ + AP + AR).$$

Comparing the two expressions gives $AQ + AP + AR = 5$. Since O lies on segment AP, $AO = AP - OP = AP - 1$, and

$$AO + AQ + AR = (AP + AQ + AR) - 1 = 4.$$

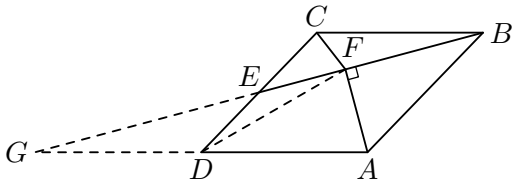
Answer: (C) 4. □

14. (2022 AMC 10B, Problem #20) Let $ABCD$ be a rhombus with $\angle ADC = 46°$. Let E be the midpoint of $\overline{CD}$, and let F be the point on $\overline{BE}$ such that $\overline{AF}$ is perpendicular to $\overline{BE}$. What is the degree measure of $\angle BFC$?

(A) 110 **(B)** 111 **(C)** 112 **(D)** 113 **(E)** 114

Strategy. Let lines BE and AD cross at G. Then D is the midpoint of the hypotenuse AG of the right triangle AFG, so D is equally far from A, F, C and G, and two isosceles triangles with the common vertex D give the angles at F.

Solution. Let G be the point where line BE, continued past E, crosses line AD, as in the figure.



In triangles ECB and EDG, the sides EC and ED are equal because E is the midpoint of CD, the angles at E are vertical angles, and $\angle ECB = \angle EDG$ because $BC \parallel AD$. So the two triangles are congruent (ASA), and $DG = CB = DA$. Since $AB \parallel CD$, the points A and B lie on the same side of line CD, and G lies on the other side, because segment BG crosses CD at E. So D lies between A and G, and D is the midpoint of AG.

The segment AF is perpendicular to BE, and G lies on line BE, so triangle AFG has a right angle at F and hypotenuse AG. By the midpoint-of-the-hypotenuse theorem, $DF = DA = DG$. The sides of the rhombus give $DC = DA$ as well, so

$$DF = DC = DG.$$

Let $y = \angle FDC$. The point F lies inside the rhombus, so $\angle ADF = 46° - y$, and since A, D, G lie on a line, $\angle FDG = 180° - \angle ADF = 134° + y$. The isosceles triangles DFC and DFG have the base angles

$$\angle DFC = \frac{180° - y}{2} = 90° - \frac{y}{2} \quad \text{and} \quad \angle DFG = \frac{180° - (134° + y)}{2} = 23° - \frac{y}{2}.$$

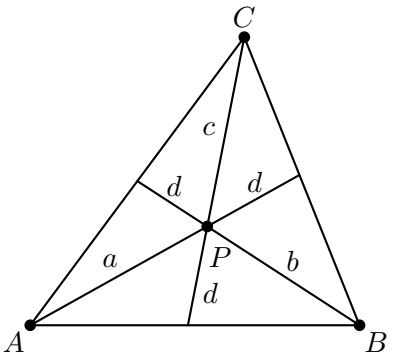
On line BG the points come in the order B, F, E, G. The point E lies on segment CD, so the ray FG, which passes through E, lies inside $\angle CFD$, and

$$\angle CFG = \angle DFC - \angle DFG = \left(90° - \frac{y}{2}\right) - \left(23° - \frac{y}{2}\right) = 67°.$$

Finally, $\angle BFC$ and $\angle CFG$ together form a straight angle, so $\angle BFC = 180° - 67° = 113°$.

Answer: (D) 113. □

15. (1988 AIME, Problem #12) Let P be an interior point of triangle ABC and extend lines from the vertices through P to the opposite sides. Let a, b, c, and d denote the lengths of the segments indicated in the figure. Find the product abc if $a+b+c=43$ and $d=3$.



Strategy. On the cevian from A, the ratio $d:(a+d)$ equals $[PBC]:[ABC]$, by the equal-heights theorem used with the vertices B and C. The three such ratios add up to 1, which gives an equation in a, b and c.

Solution. Let the lines from A, B and C through P meet the opposite sides at D, E and F. As in the figure, $AP=a$, $BP=b$, $CP=c$ and $PD=PE=PF=d$.

Triangles PBD and ABD have the common vertex B and their bases PD and AD on line AD, and triangles PCD and ACD have the common vertex C and their bases on the same line. By the equal-heights theorem,

$$[PBD]=\frac{PD}{AD}\,[ABD] \qquad \text{and} \qquad [PCD]=\frac{PD}{AD}\,[ACD].$$

Adding the two equations,

$$[PBC]=\frac{PD}{AD}\,[ABC], \qquad \text{that is,} \qquad \frac{d}{a+d}=\frac{[PBC]}{[ABC]}.$$

The same argument for the other two cevians gives

$$\frac{d}{b+d}=\frac{[PCA]}{[ABC]} \qquad \text{and} \qquad \frac{d}{c+d}=\frac{[PAB]}{[ABC]}.$$

The triangles PBC, PCA and PAB make up triangle ABC, so

$$\frac{d}{a+d}+\frac{d}{b+d}+\frac{d}{c+d}=1.$$

With $d=3$, multiplying by $(a+3)(b+3)(c+3)$ gives

$$3\big[(b+3)(c+3)+(c+3)(a+3)+(a+3)(b+3)\big]=(a+3)(b+3)(c+3).$$

The left side is $3(ab+bc+ca)+18(a+b+c)+81$, and the right side is $abc+3(ab+bc+ca)+9(a+b+c)+27$. Hence

$$abc=9(a+b+c)+54=9\cdot 43+54=441.$$

As a check, $a=b=21$ and $c=1$ satisfy $a+b+c=43$ and

$$\frac{3}{24}+\frac{3}{24}+\frac{3}{4}=1,$$

and their product is $21\cdot 21\cdot 1=441$.

Answer: 441. □

2. Answer Key

Problem	Letter	Value
1	C	$\frac{5}{2}$
2	C	$5\frac{5}{7}$
3	D	50
4	A	$1:3$
5	D	$\frac{1}{12}$
6	A	$5:4$
7	A	$\frac{1}{5}$
8	D	13
9	B	$\frac{3\sqrt{3}}{\pi}$
10	C	911
11	D	$\frac{8}{15}\sqrt{5}$
12	B	$5:11$
13	C	4
14	D	113
15		441