

ANGLE CHASING AND POLYGONS FOR AMC 10

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1. ANGLE FACTS

Angle chasing finds the unknown angles in a figure one step at a time, and each step uses a basic fact about lines, parallel lines or triangles. This section collects these facts.

1.1. Lines and Parallel Lines. Two angles are **complementary** if their measures add to 90° and **supplementary** if they add to 180° . Three facts are used throughout:

- Angles that together form a straight line add to 180° .
- Angles that together go all the way around a point add to 360° .
- When two lines cross, the opposite angles, called **vertical angles**, are equal.

A line that crosses two other lines is a **transversal**. It makes four angles with each of the two lines.

Theorem 1.1.1 (Parallel lines). *Suppose a transversal crosses two parallel lines. Then*

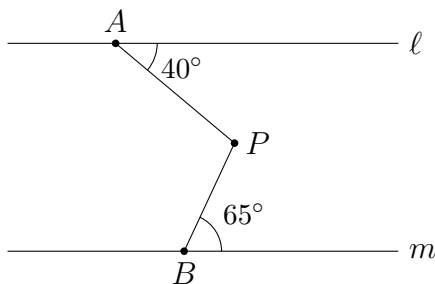
- corresponding angles are equal,*
- alternate interior angles are equal, and*
- the two interior angles on the same side of the transversal are supplementary.*

Conversely, if one of these three statements holds for a transversal of two lines, the two lines are parallel.

When a figure contains parallel lines but no transversal passes through the point of interest, draw a new line through that point parallel to the given ones. The next example uses this idea.

Example.

Problem 1.1.2. In the figure, lines ℓ and m are parallel, point A lies on ℓ , point B lies on m , and point P lies between the two lines. Segment AP makes a 40° angle with ℓ and segment BP makes a 65° angle with m , as marked. Find $\angle APB$.



Strategy. Draw the line through P parallel to ℓ . Segments AP and BP then give two pairs of alternate interior angles.

Solution. Let n be the line through P parallel to ℓ . Then n is also parallel to m , and it splits $\angle APB$ into an upper part, between PA and n , and a lower part, between n and PB .

Segment AP is a transversal of the parallel lines ℓ and n . The upper part of $\angle APB$ and the marked angle at A are alternate interior angles, so the upper part is 40° . In the same way, BP is a transversal of the parallel lines n and m , so the lower part equals the marked angle at B , which is 65° . Therefore

$$\angle APB = 40^\circ + 65^\circ = 105^\circ.$$

The same argument proves a general rule: when a path bends at a point between two parallel lines, as in this figure, the angle at the bend is the sum of the angles that the two segments make with the parallel lines.

Answer: 105° . □

1.2. Angles of a Triangle.

Theorem 1.2.1 (Triangle angle sum). *The three angles of a triangle add to 180° .*

Proof. In triangle ABC , draw the line through A parallel to BC . At A , this line and the sides AB and AC form three angles that together make a straight line. The outer two are alternate interior angles with $\angle B$ and $\angle C$, and the middle one is $\angle A$. So $\angle A + \angle B + \angle C = 180^\circ$. $\square$

An **exterior angle** of a triangle is the angle between one side and the extension of another side, such as $\angle ACD$ when side BC is extended past C to a point D .

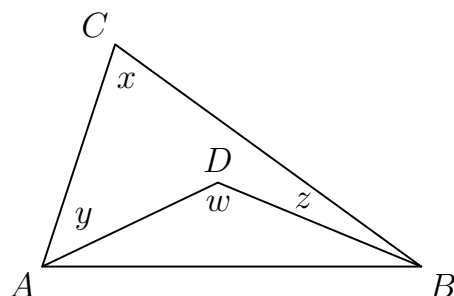
Theorem 1.2.2 (Exterior angle). *An exterior angle of a triangle equals the sum of the interior angles at the other two vertices. If D lies on line BC beyond C , then $\angle ACD = \angle A + \angle B$.*

Proof. The angles $\angle ACD$ and $\angle ACB$ form a straight line, so $\angle ACD = 180^\circ - \angle ACB = \angle A + \angle B$ by the angle sum. $\square$

The exterior angle theorem shortens many angle chases, because it gives the exterior angle without the third angle of the triangle. A quadrilateral splits along a diagonal into two triangles, so its angles add to 360° ; Section 4 extends this to every polygon.

Example.

Problem 1.2.3 (1987 AHSME, Problem #6). In the $\triangle ABC$ shown, D is some interior point, and x, y, z, w are the measures of angles in degrees. Solve for x in terms of y, z , and w .



- (A) $w - y - z$ (B) $w - 2y - 2z$ (C) $180 - w - y - z$ (D) $2w - y - z$
 (E) $180 - w + y + z$

Strategy. Extend AD until it meets BC . Then w and the new angle on BC are exterior angles of two triangles.

Solution. Extend AD beyond D to meet side BC at a point E . In triangle AEC , the angle $\angle AEB$ is an exterior angle at E , so

$$\angle AEB = \angle EAC + \angle ACE = y + x.$$

In triangle DEB , the angle $w = \angle ADB$ is an exterior angle at D , so

$$w = \angle DEB + \angle DBE = (x + y) + z.$$

Hence $x = w - y - z$.

The same argument proves a general rule: in a dart-shaped quadrilateral such as $ACBD$, the angle w at the dent, measured on the outside of the dart, equals the sum $x + y + z$ of the other three angles.

Answer: (A) $w - y - z$. $\square$

2. ISOSCELES TRIANGLES

Equal sides in a figure give equal angles. Most problems that combine angles with equal lengths are solved by finding every isosceles triangle in the figure and writing its angles in terms of one unknown.

2.1. Base Angles.

Theorem 2.1.1 (Isosceles triangle). *In triangle ABC , $AB = AC$ if and only if $\angle B = \angle C$.*

The two equal angles are the **base angles**, and the third angle is the **vertex angle**. If the vertex angle is θ , each base angle is

$$\frac{180^\circ - \theta}{2} = 90^\circ - \frac{\theta}{2},$$

and if each base angle is β , the vertex angle is $180^\circ - 2\beta$. All three angles of an equilateral triangle are 60° , and an isosceles triangle with one 60° angle is equilateral.

Cutting an equilateral triangle in half along a line of symmetry gives a right triangle with angles 30° , 60° and 90° . This gives a fact about right triangles that angle chases often need.

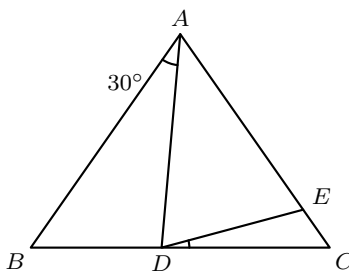
Theorem 2.1.2 (Half of an equilateral triangle). *In a right triangle, the leg opposite a 30° angle is half the hypotenuse. Conversely, if one leg of a right triangle is half the hypotenuse, the angle opposite that leg is 30° .*

Proof. Let triangle ABC have its right angle at C , and reflect B across line AC to a point B' . Then B, C, B' lie on a line, $BB' = 2BC$ and $AB' = AB$, so triangle ABB' is isosceles with vertex angle $\angle BAB' = 2\angle BAC$.

If $\angle BAC = 30^\circ$, then triangle ABB' is isosceles with a 60° angle, so it is equilateral and $2BC = BB' = AB$. If instead $2BC = AB$, then $BB' = AB = AB'$, so triangle ABB' is equilateral, $2\angle BAC = 60^\circ$, and $\angle BAC = 30^\circ$. $\square$

Example.

Problem 2.1.3 (1956 AHSME, Problem #24). In the figure $AB = AC$, angle $BAD = 30^\circ$, and $AE = AD$. Then angle CDE equals:



- (A) $7\frac{1}{2}^\circ$ (B) 10° (C) $12\frac{1}{2}^\circ$ (D) 15° (E) 20°

Strategy. The angles of triangle ABC are not given, so name its base angle b and the unknown angle x . Then write $\angle AED$ in two ways.

Solution. Let $\angle B = \angle C = b$ and $\angle CDE = x$.

The angle $\angle ADC$ is an exterior angle of triangle ABD , so

$$\angle ADC = \angle B + \angle BAD = b + 30^\circ,$$

and $\angle ADE = \angle ADC - \angle CDE = b + 30^\circ - x$. Triangle ADE is isosceles with $AD = AE$, so $\angle AED = \angle ADE = b + 30^\circ - x$.

On the other hand, $\angle AED$ is an exterior angle of triangle DEC , so

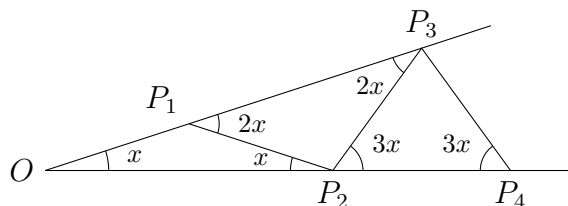
$$\angle AED = \angle EDC + \angle ECD = x + b.$$

Setting the two expressions equal gives $b + 30^\circ - x = x + b$, so $x = 15^\circ$. The base angle b cancels, so the answer does not depend on the shape of triangle ABC : the angle $\angle CDE$ is always half of $\angle BAD$.

Answer: (D) 15° . □

2.2. Isosceles Chains. A chain of equal segments, such as $AB = BC = CD$, gives a chain of isosceles triangles in which consecutive triangles share a side. The exterior angle theorem carries an angle from each triangle to the next. The method is always the same: call the smallest angle x , walk along the chain writing each new angle in terms of x , and finish with an angle sum.

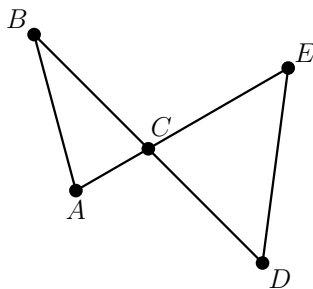
The basic pattern is the zigzag below. Two rays from O form an angle x , and the points P_1, P_2, P_3, P_4 lie alternately on the two rays with $OP_1 = P_1P_2 = P_2P_3 = P_3P_4$.



Triangle OP_1P_2 is isosceles with base angles x . Its exterior angle at P_1 is $2x$, and this is a base angle of the isosceles triangle $P_1P_2P_3$. The exterior angle of triangle OP_2P_3 at P_2 is then $x + 2x = 3x$, which is a base angle of the isosceles triangle $P_2P_3P_4$. In general, the base angles of the successive triangles are $x, 2x, 3x$, and so on.

Examples.

Problem 2.2.1 (2009 AMC 10B, Problem #9). Segment BD and AE intersect at C , as shown, $AB = BC = CD = CE$, and $\angle A = \frac{5}{2}\angle B$. What is the degree measure of $\angle D$?



- (A) 52.5 (B) 55 (C) 57.5 (D) 60 (E) 62.5

Strategy. The equal segments give two isosceles triangles, ABC and CDE . Find the angles of the first one, then pass to the second through the vertical angles at C .

Solution. Since $AB = BC$, triangle ABC is isosceles with $\angle BCA = \angle A$. Let $\angle B = b$, so that $\angle A = \angle BCA = \frac{5}{2}b$. The angle sum in triangle ABC gives

$$b + \frac{5}{2}b + \frac{5}{2}b = 6b = 180^\circ,$$

so $b = 30^\circ$ and $\angle BCA = 75^\circ$.

The angles $\angle BCA$ and $\angle DCE$ are vertical angles, so $\angle DCE = 75^\circ$. Since $CD = CE$, triangle CDE is isosceles with vertex angle 75° , and

$$\angle D = \frac{180^\circ - 75^\circ}{2} = 52.5^\circ.$$

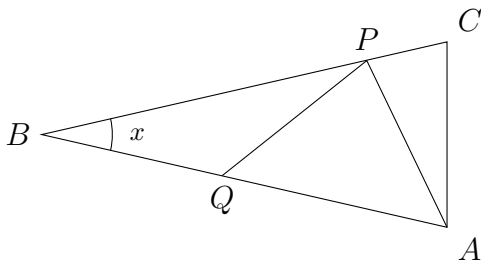
Answer: (A) 52.5. □

Problem 2.2.2 (1961 AHSME, Problem #25). Triangle ABC is isosceles with base AC . Points P and Q are respectively in CB and AB and such that $AC = AP = PQ = QB$. The number of degrees in angle B is:

- (A) $25\frac{5}{7}$ (B) $26\frac{1}{3}$ (C) 30 (D) 40 (E) not determined by the information given

Strategy. This is the zigzag pattern with vertex B . Let $\angle B = x$ and walk along the chain $QB = PQ = AP = AC$, starting at B .

Solution. The figure shows the triangle with B on the left.



Let $\angle B = x$.

- Since $QB = QP$, triangle BQP is isosceles and $\angle QPB = \angle QBP = x$.
- The angle $\angle AQP$ is an exterior angle of triangle BQP , so $\angle AQP = 2x$. Since $PQ = PA$, triangle PQA is isosceles and $\angle QAP = \angle AQP = 2x$.
- The angle $\angle APC$ is an exterior angle of triangle ABP , so $\angle APC = \angle PAB + \angle PBA = 2x + x = 3x$. Since $AP = AC$, triangle APC is isosceles and $\angle ACP = \angle APC = 3x$.

Triangle ABC is isosceles with base AC , so $\angle BAC = \angle BCA = 3x$. The angle sum in triangle ABC gives $x + 3x + 3x = 180^\circ$, so

$$x = \frac{180^\circ}{7} = 25\frac{5}{7}^\circ.$$

Answer: (A) $25\frac{5}{7}$. □

2.3. Auxiliary Constructions. Sometimes the given equal segments do not form any isosceles triangle. Then add a point that creates new equal segments. A common choice is the third vertex of an equilateral triangle built on one of the given segments: it is at the same distance from both ends of that segment, and it brings in 60° angles. To show that three points lie on a line, show that the angles at the middle point add to 180° .

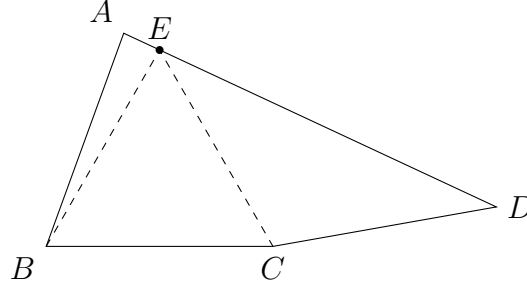
Example.

Problem 2.3.1 (2008 AMC 10B, Problem #24). Quadrilateral $ABCD$ has $AB = BC = CD$, $\angle ABC = 70^\circ$, and $\angle BCD = 170^\circ$. What is the degree measure of $\angle BAD$?

- (A) 75 (B) 80 (C) 85 (D) 90 (E) 95

Strategy. The segments AB , BC , CD share endpoints but form no triangle. Build the equilateral triangle BCE on BC , inside the quadrilateral. Then E is at distance BC from both B and C , which makes two isosceles triangles.

Solution. Let E be the point on the same side of BC as A and D such that triangle BCE is equilateral (dashed in the figure). Then $BE = BC = BA$ and $CE = CB = CD$.



Since $\angle ABC = 70^\circ$ and $\angle EBC = 60^\circ$, the angle $\angle ABE$ is $70^\circ - 60^\circ = 10^\circ$. Triangle ABE is isosceles with $BA = BE$ and vertex angle 10° , so

$$\angle BAE = \angle BEA = \frac{180^\circ - 10^\circ}{2} = 85^\circ.$$

Similarly, $\angle ECD = 170^\circ - 60^\circ = 110^\circ$, and triangle CED is isosceles with $CE = CD$, so

$$\angle CED = \angle CDE = \frac{180^\circ - 110^\circ}{2} = 35^\circ.$$

The three angles at E between A and D add to

$$\angle AEB + \angle BEC + \angle CED = 85^\circ + 60^\circ + 35^\circ = 180^\circ,$$

so A , E , D lie on a line, with E between A and D . Therefore $\angle BAD = \angle BAE = 85^\circ$.

Answer: (C) 85.

□

3. ANGLE BISECTORS

The **bisector** of an angle is the ray that splits it into two equal angles. Bisectors produce half-angles, and a bisector together with a parallel line produces an isosceles triangle. At each vertex of a triangle, the bisectors of the interior angle and of the exterior angle are perpendicular: if the interior angle is θ , the exterior angle is $180^\circ - \theta$, and the two halves add to

$$\frac{\theta}{2} + \frac{180^\circ - \theta}{2} = 90^\circ.$$

3.1. Angles Between Bisectors. In a triangle ABC , the letters A , B , C also stand for the angles at those vertices.

Theorem 3.1.1. *Let ABC be a triangle.*

- (a) *The bisectors of the interior angles at B and C meet at a point P with $\angle BPC = 90^\circ + \frac{A}{2}$.*
- (b) *The bisectors of the exterior angles at B and C meet at a point Q with $\angle BQC = 90^\circ - \frac{A}{2}$.*
- (c) *The bisector of the interior angle at B and the bisector of the exterior angle at C meet at a point R with $\angle BRC = \frac{A}{2}$.*

Proof. (a) In triangle BPC ,

$$\angle BPC = 180^\circ - \frac{B}{2} - \frac{C}{2} = 180^\circ - \frac{180^\circ - A}{2} = 90^\circ + \frac{A}{2}.$$

(b) This is the next example.

(c) Extend BC beyond C to a point X . The exterior angle at C is $\angle ACX = A + B$, so $\angle RCX = \frac{A+B}{2}$. This is also an exterior angle of triangle BRC , so $\angle RCX = \angle RBC + \angle BRC = \frac{B}{2} + \angle BRC$. Comparing the two expressions gives $\angle BRC = \frac{A}{2}$. $\square$

Example.

Problem 3.1.2 (1962 AHSME, Problem #7). Let the bisectors of the exterior angles at B and C of triangle ABC meet at D . Then, if all measurements are in degrees, angle BDC equals:

- (A) $\frac{1}{2}(90 - A)$ (B) $90 - A$ (C) $\frac{1}{2}(180 - A)$ (D) $180 - A$ (E) $180 - 2A$

Strategy. The angles of triangle BDC at B and C are halves of exterior angles. Write them in terms of B and C , and use the angle sum in triangle BDC .

Solution. The exterior angle at B between side BC and the extension of side AB is $180 - B$, and the bisector cuts it in half, so $\angle DBC = 90 - \frac{B}{2}$. In the same way, $\angle DCB = 90 - \frac{C}{2}$. The angle sum in triangle BDC gives

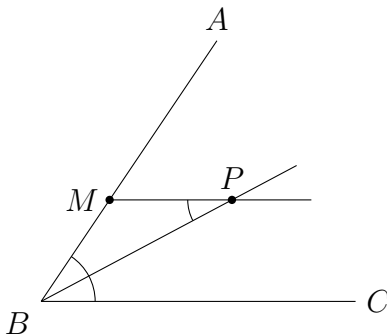
$$\angle BDC = 180 - \left(90 - \frac{B}{2}\right) - \left(90 - \frac{C}{2}\right) = \frac{B + C}{2} = \frac{180 - A}{2}.$$

This equals $90 - \frac{A}{2}$, which proves part (b) of the theorem.

Answer: (C) $\frac{1}{2}(180 - A)$. $\square$

3.2. Bisectors and Parallel Lines.

Theorem 3.2.1. Let $P \neq B$ be a point on the bisector of $\angle ABC$, and let the line through P parallel to BC meet line BA at M . Then $MB = MP$.

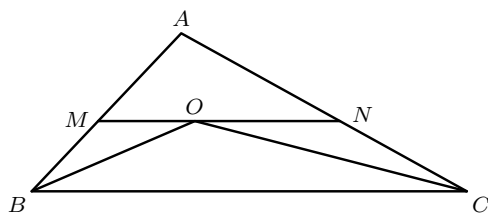


Proof. Since $MP \parallel BC$, the alternate interior angles $\angle MPB$ and $\angle PBC$ are equal. Since BP bisects $\angle ABC$, $\angle PBC = \angle MBP$. So $\angle MPB = \angle MBP$, and triangle MBP is isosceles with $MB = MP$. $\square$

For the same reason, in a parallelogram $ABCD$ the bisector of $\angle A$ meets line CD at a point E with $DE = DA$.

Example.

Problem 3.2.2 (1982 AHSME, Problem #10). In the adjoining diagram, BO bisects $\angle CBA$, CO bisects $\angle ACB$, and MN is parallel to BC . If $AB = 12$, $BC = 24$, and $AC = 18$, then the perimeter of $\triangle AMN$ is



- (A) 30 (B) 33 (C) 36 (D) 39 (E) 42

Strategy. Each bisector meets the parallel line MN , so the theorem gives two isosceles triangles, MBO and NCO . Use them to replace MO and ON in the perimeter.

Solution. By the theorem at vertex B , triangle MBO is isosceles with $MB = MO$. By the theorem at vertex C , triangle NCO is isosceles with $NC = NO$. Therefore

$$\begin{aligned} AM + MN + NA &= AM + MO + ON + NA \\ &= (AM + MB) + (AN + NC) = AB + AC = 12 + 18 = 30. \end{aligned}$$

The length $BC = 24$ is not needed.

Answer: (A) 30. □

4. POLYGON ANGLE SUMS

A polygon with n sides is an n -**gon**. It is **convex** if each of its interior angles is less than 180° .

4.1. Interior Angles.

Theorem 4.1.1. The interior angles of a convex n -gon add to $(n - 2) \cdot 180^\circ$.

Proof. The diagonals from one vertex cut the n -gon into $n - 2$ triangles. Together, the angles of these triangles make up exactly the angles of the polygon, so the sum is $(n - 2) \cdot 180^\circ$. □

The formula also holds for a polygon that is not convex, as long as its sides do not cross each other. At a vertex where such a polygon is dented inward, the interior angle is a **reflex angle**, an angle larger than 180° .

Example.

Problem 4.1.2 (1982 AHSME, Problem #6). The sum of all but one of the interior angles of a convex polygon equals 2570° . The remaining angle is

- (A) 90° (B) 105° (C) 120° (D) 130° (E) 144°

Strategy. The full sum is a multiple of 180° . The remaining angle is between 0° and 180° , because the polygon is convex, so the full sum is the first multiple of 180° above 2570° .

Solution. Let the polygon have n sides and let x be the remaining angle, in degrees. Then

$$(n - 2) \cdot 180 = 2570 + x, \quad 0 < x < 180,$$

so the full sum is a multiple of 180 strictly between 2570 and 2750. Since $180 \cdot 14 = 2520$ and $180 \cdot 15 = 2700$, the only such multiple is 2700. Hence $n - 2 = 15$, the polygon has 17 sides, and $x = 2700 - 2570 = 130$.

Answer: (D) 130° . □

4.2. Exterior Angles. At a vertex of a convex polygon, the **exterior angle** is the angle between one side and the extension of the adjacent side. It equals 180° minus the interior angle at that vertex.

Theorem 4.2.1. *The exterior angles of a convex polygon, one at each vertex, add to 360° .*

Proof. At each of the n vertices, the interior and exterior angles add to 180° , so all $2n$ angles together add to $n \cdot 180^\circ$. Subtracting the interior sum $(n-2) \cdot 180^\circ$ leaves 360° . $\square$

Equivalently, a walk once around the polygon turns through the exterior angle at each vertex and ends facing its original direction, so the turns add to one full turn.

Unlike the interior sum, the exterior sum does not depend on n . This makes exterior angles the right tool for questions about how many angles of a polygon can be large or small. For example, an interior angle is obtuse exactly when its exterior angle is acute.

Example.

Problem 4.2.2 (1985 AHSME, Problem #14). Exactly three of the interior angles of a convex polygon are obtuse. What is the maximum number of sides of such a polygon?

(A) 4 (B) 5 (C) 6 (D) 7 (E) 8

Strategy. Pass to exterior angles: three of them are acute, every other one is at least 90° , and all of them together add to only 360° .

Solution. Suppose the polygon has n sides. Each of the $n-3$ interior angles that are not obtuse is at most 90° , so its exterior angle is at least 90° . The other three exterior angles are positive. Hence

$$360^\circ > (n-3) \cdot 90^\circ,$$

so $n-3 < 4$ and $n \leq 6$.

Six sides are possible. Cut off each corner of an equilateral triangle by a short segment perpendicular to one of the two sides at that corner. Each cut replaces a 60° angle by a 90° angle and a 150° angle, so the resulting hexagon has three right angles and exactly three obtuse angles.

Answer: (C) 6. $\square$

4.3. Regular Polygons. A polygon is **equiangular** if all of its angles are equal, **equilateral** if all of its sides are equal, and **regular** if it is both. In an equiangular n -gon, and in particular in a regular one, the n exterior angles are equal, so

$$\text{each exterior angle} = \frac{360^\circ}{n}, \quad \text{each interior angle} = 180^\circ - \frac{360^\circ}{n} = \frac{(n-2) \cdot 180^\circ}{n}.$$

Number of sides n	3	4	5	6	8	9	10	12
Each interior angle	60°	90°	108°	120°	135°	140°	144°	150°
Each exterior angle	120°	90°	72°	60°	45°	40°	36°	30°

Three more facts are useful.

- A regular n -gon has a **center** O at the same distance from every vertex. The segments from O to the vertices cut the polygon into n congruent isosceles triangles, each with vertex angle $\frac{360^\circ}{n}$ at O .
- Two adjacent sides of a regular n -gon and the diagonal joining their far ends form an isosceles triangle whose vertex angle is an interior angle of the polygon. Its base angles are $\frac{180^\circ}{n}$. In a regular pentagon $ABCDE$, for example, $\angle BAC = \angle EAD = 36^\circ$, so the diagonals AC and AD split the 108° angle at A into three 36° angles.

- When several polygons fit around a point with no gaps or overlaps, their angles at that point add to 360° .

Examples.

Problem 4.3.1 (1993 AHSME, Problem #15). For how many values of n will an n -sided regular polygon have interior angles with integral degree measures?

(A) 16 (B) 18 (C) 20 (D) 22 (E) 24

Strategy. Write the interior angle as $180 - \frac{360}{n}$. It is an integer exactly when $\frac{360}{n}$ is.

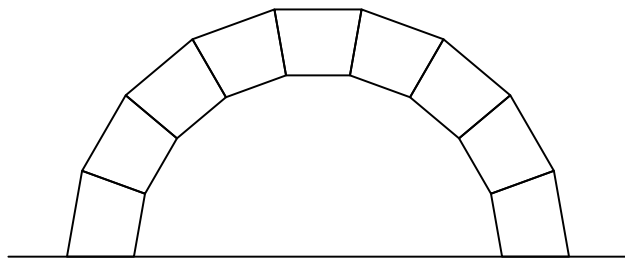
Solution. Each interior angle of a regular n -gon measures $180 - \frac{360}{n}$ degrees, which is an integer exactly when n divides 360. The divisors of 360 come in pairs with product 360:

$1 \cdot 360, 2 \cdot 180, 3 \cdot 120, 4 \cdot 90, 5 \cdot 72, 6 \cdot 60, 8 \cdot 45, 9 \cdot 40, 10 \cdot 36, 12 \cdot 30, 15 \cdot 24, 18 \cdot 20,$

so 360 has 24 divisors. (This agrees with the count from the exponents: $360 = 2^3 \cdot 3^2 \cdot 5$ has $4 \cdot 3 \cdot 2 = 24$ divisors.) A polygon has at least 3 sides, so $n = 1$ and $n = 2$ do not count, which leaves $24 - 2 = 22$ values.

Answer: (D) 22. □

Problem 4.3.2 (2009 AMC 10B, Problem #24). The keystone arch is an ancient architectural feature. It is composed of congruent isosceles trapezoids fitted together along the non-parallel sides, as shown. The bottom sides of the two end trapezoids are horizontal. In an arch made with 9 trapezoids, let x be the angle measure in degrees of the larger interior angle of the trapezoid. What is x ?



(A) 100 (B) 102 (C) 104 (D) 106 (E) 108

Strategy. Extend the non-parallel sides of the trapezoids until they meet. The arch becomes a fan of congruent isosceles triangles around one point, like the triangles around the center of a regular polygon.

Solution. Extend the two legs (the non-parallel sides) of each trapezoid until they meet. Each trapezoid becomes an isosceles triangle with its top cut off. Neighboring trapezoids are congruent and share a leg, so all of these triangles have the same vertex O , the center of the arch. The two end legs are horizontal and both pass through O , so they lie on one line, on opposite sides of O . Hence the 9 equal vertex angles at O fill a straight angle, and each is $\frac{180^\circ}{9} = 20^\circ$.

The base angles of each triangle are therefore $\frac{180^\circ - 20^\circ}{2} = 80^\circ$. These are the smaller angles of the trapezoid, at its longer base. The two bases of a trapezoid are parallel, so the angles at the two ends of a leg are supplementary, and the larger angle is $x = 180 - 80 = 100$.

As a check, the inner edge of the arch is half of a regular 18-gon centered at O , since $20^\circ = \frac{360^\circ}{18}$. Its interior angles are 160° , and at each inner corner the two trapezoid angles and the 160° angle go around the point: $2x + 160 = 360$, so $x = 100$.

Answer: (A) 100. □

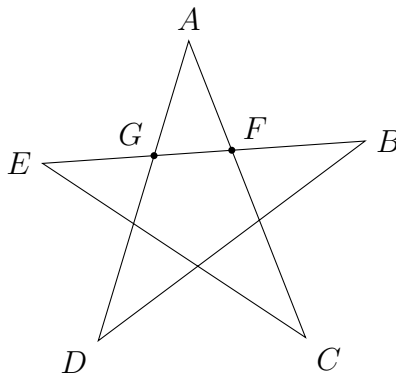
5. STAR POLYGONS

A **star polygon** is a closed path of segments that crosses itself, such as the five-pointed star. The angle at each tip of the star is a **point angle**. Point angles are found with the exterior angle theorem and with the angle sums of Section 4.

5.1. Five-Pointed Stars.

Theorem 5.1.1. *In every five-pointed star, the five point angles add to 180° .*

Proof. Label the points A, B, C, D, E in order around the star, so that the segments of the star are AC, CE, EB, BD and DA . Segment BE crosses AC at F and AD at G .



Since A, F, C lie on a line, $\angle AFG$ is an exterior angle of triangle FCE , so it equals the sum of the point angles at C and E . Since A, G, D lie on a line, $\angle AGF$ is an exterior angle of triangle GDB , so it equals the sum of the point angles at D and B . The angle sum in triangle AFG now gives

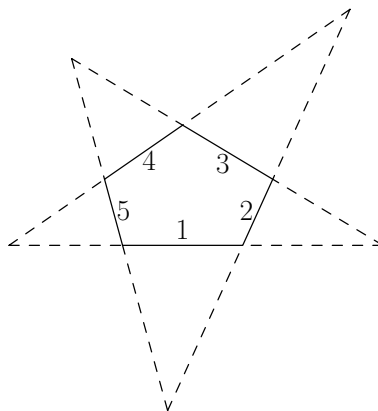
$$\angle A + (\angle C + \angle E) + (\angle D + \angle B) = 180^\circ. \quad \square$$

In a regular five-pointed star the five point angles are equal, so each is 36° . This agrees with Section 4: the diagonals of a regular pentagon form a regular five-pointed star, and the angle between the two diagonals at a vertex is 36° .

5.2. Stars from Extended Sides. Extending the sides of a convex polygon can produce a star. Number the sides in order around the polygon. Suppose the exterior angles at the two ends of each side add to less than 180° . Then sides k and $k + 2$, extended beyond side $k + 1$, meet at a point outside the polygon. That point is the top vertex of a triangle whose base is side $k + 1$. The condition on the exterior angles is needed: convexity alone does not guarantee this configuration.

Example.

Problem 5.2.1 (Adapted from 1966 AHSME, Problem #21). An “ n -pointed star” is formed as follows: the sides of a convex polygon are numbered consecutively $1, 2, \dots, k, \dots, n$, $n \geq 5$; for all n values of k , sides k and $k + 2$ are non-parallel, sides $n + 1$ and $n + 2$ being respectively identical with sides 1 and 2; prolong the n pairs of sides numbered k and $k + 2$ until they meet. (A figure is shown for the case $n = 5$.)



Assume also that the exterior angles at the two ends of every side of the polygon add to less than 180° . Let S be the degree-sum of the interior angles at the n points of the star; then S equals:

- (A) 180 (B) 360 (C) $180(n + 2)$ (D) $180(n - 2)$ (E) $180(n - 4)$

Strategy. Each point of the star is the top of a triangle whose base is a side of the polygon. Its base angles are exterior angles of the polygon, and the exterior angles add to 360° .

Solution. Consider the point formed by sides k and $k + 2$. Together with side $k + 1$ they form a triangle with base on side $k + 1$. The base angles of this triangle are the angles between side $k + 1$ and the extensions of sides k and $k + 2$, so they are the exterior angles of the polygon at the two ends of side $k + 1$. If these exterior angles are e and e' , the point angle is $180^\circ - e - e'$.

Now add the n point angles. Every vertex of the polygon is an end of exactly two sides, so every exterior angle is subtracted exactly twice:

$$S = 180n - 2 \cdot 360 = 180(n - 4).$$

For $n = 5$ this gives 180, in agreement with the theorem on five-pointed stars.

Answer: (E) $180(n - 4)$. □

5.3. Regular Star Polygons. Place n points equally spaced around a circle with center O , and join each point to the point k places further around, where $2 \leq k < \frac{n}{2}$. The result is a **regular star polygon**. (If n and k have a common factor, the path closes before it reaches every point, and the figure consists of several separate closed paths; the point angles are still given by the theorem below.) The proof below uses only two facts about the circle: O is at the same distance from all n points, and the **central angle** $\angle POQ$ between two neighboring points P and Q is $\frac{360^\circ}{n}$.

Theorem 5.3.1. *Each point angle of this regular star polygon is*

$$180^\circ - \frac{360^\circ k}{n} = \frac{(n - 2k) \cdot 180^\circ}{n}.$$

Proof. Let P be one of the points, and let Q and R be the points k places away from P on either side, so that the two segments of the star at P are PQ and PR . The central angle $\angle POQ$ is

$k \cdot \frac{360^\circ}{n}$. Since $OP = OQ$, triangle POQ is isosceles, and its base angle is

$$\angle OPQ = 90^\circ - \frac{180^\circ k}{n}.$$

In the same way, $\angle OPR = 90^\circ - \frac{180^\circ k}{n}$. By symmetry the ray PO lies between PQ and PR , so the point angle is $\angle QPR = 180^\circ - \frac{360^\circ k}{n}$. $\square$

For the regular five-pointed star, $n = 5$ and $k = 2$, and the formula gives $180^\circ - 144^\circ = 36^\circ$, as before. Adding the n point angles gives a total of $180n - 360k$ degrees.

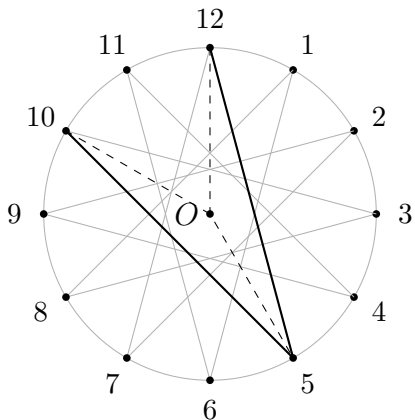
Example.

Problem 5.3.2 (2007 AMC 12A, Problem #8). A star-polygon is drawn on a clock face by drawing a chord from each number to the fifth number counted clockwise from that number. That is, chords are drawn from 12 to 5, from 5 to 10, from 10 to 3, and so on, ending back at 12. What is the degree measure of the angle at each vertex in the star-polygon?

(A) 20 (B) 24 (C) 30 (D) 36 (E) 60

Strategy. Use the center O of the clock. The two chords at a vertex are bases of isosceles triangles with vertex O .

Solution. Let O be the center of the clock face. Neighboring numbers are $\frac{360^\circ}{12} = 30^\circ$ apart as seen from O . In the figure, the two chords at the vertex 5 are drawn thick, and the segments from O to 12, 5 and 10 are dashed.



Look at the vertex 5. Its two chords go to 12 and to 10, each five numbers away, so $\angle 5O12 = \angle 5O10 = 5 \cdot 30^\circ = 150^\circ$. Triangle $5O12$ is isosceles with $O5 = O12$, so its base angle at 5 is $\frac{180^\circ - 150^\circ}{2} = 15^\circ$, and the same holds for triangle $5O10$. The segment from 5 to O lies between the two chords, so the angle at vertex 5 is $15^\circ + 15^\circ = 30^\circ$. The same computation applies at every vertex, so every angle of the star is 30° .

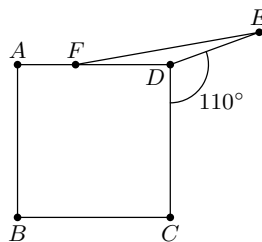
This agrees with the theorem for $n = 12$ and $k = 5$: $180^\circ - \frac{360^\circ \cdot 5}{12} = 30^\circ$.

Answer: (C) 30. $\square$

6. PROBLEMS

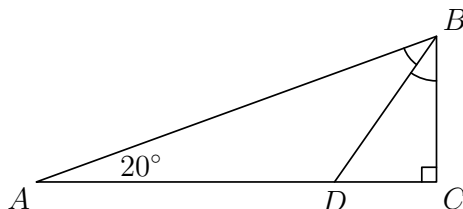
Beginner.

1. (2021 Fall AMC 10A, Problem #7) As shown in the figure below, point E lies in the opposite half-plane determined by line CD from point A so that $\angle CDE = 110^\circ$. Point F lies on $\overline{AD}$ so that $DE = DF$, and $ABCD$ is a square. What is the degree measure of $\angle AFE$?



- (A) 160 (B) 164 (C) 166 (D) 170 (E) 174

2. (1986 AHSME, Problem #3) In the figure, $\triangle ABC$ has a right angle at C and $\angle A = 20^\circ$. If BD is the bisector of $\angle ABC$, then $\angle BDC =$

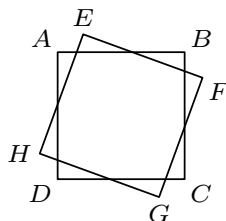


- (A) 40° (B) 45° (C) 50° (D) 55° (E) 60°

3. (2007 AMC 10B, Problem #7) All sides of the convex pentagon $ABCDE$ are of equal length, and $\angle A = \angle B = 90^\circ$. What is the degree measure of $\angle E$?

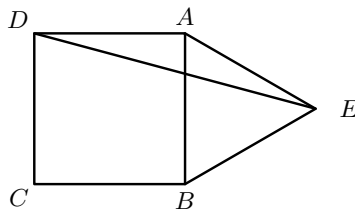
- (A) 90 (B) 108 (C) 120 (D) 144 (E) 150

4. (2023 AMC 10B, Problem #7) Square $ABCD$ is rotated 20° clockwise about its center to obtain square $EFGH$, as shown below. What is the degree measure of $\angle EAB$?



- (A) 24° (B) 35° (C) 30° (D) 32° (E) 20°

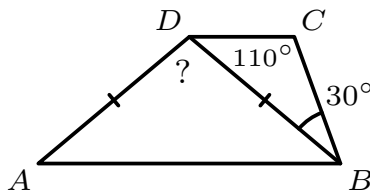
5. (1979 AHSME, Problem #3) In the adjoining figure, $ABCD$ is a square, ABE is an equilateral triangle and point E is outside square $ABCD$. What is the measure of $\angle AED$ in degrees?



- (A) 10 (B) 12.5 (C) 15 (D) 20 (E) 25

Intermediate.

6. (1981 AHSME, Problem #5) In trapezoid $ABCD$, sides AB and CD are parallel, and diagonal BD and side AD have equal length. If $\angle DCB = 110^\circ$ and $\angle CBD = 30^\circ$, then $\angle ADB =$



- (A) 80° (B) 90° (C) 100° (D) 110° (E) 120°

7. (1960 AHSME, Problem #27) Let S be the sum of the interior angles of a polygon P for which each interior angle is $7\frac{1}{2}$ times the exterior angle at the same vertex. Then:
- (A) $S = 2660^\circ$ and P may be regular (B) $S = 2660^\circ$ and P is not regular
 (C) $S = 2700^\circ$ and P is regular (D) $S = 2700^\circ$ and P is not regular
 (E) $S = 2700^\circ$ and P may or may not be regular

8. (2011 AMC 10B, Problem #18) Rectangle $ABCD$ has $AB = 6$ and $BC = 3$. Point M is chosen on side AB so that $\angle AMD = \angle CMD$. What is the degree measure of $\angle AMD$?
- (A) 15 (B) 30 (C) 45 (D) 60 (E) 75

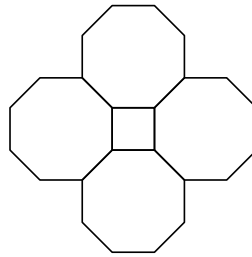
9. (1952 AHSME, Problem #21) The sides of a regular polygon of n sides, $n > 4$, are extended to form a star. The number of degrees at each point of the star is:

- (A) $\frac{360}{n}$ (B) $\frac{(n-4)180}{n}$ (C) $\frac{(n-2)180}{n}$ (D) $180 - \frac{90}{n}$ (E) $\frac{180}{n}$

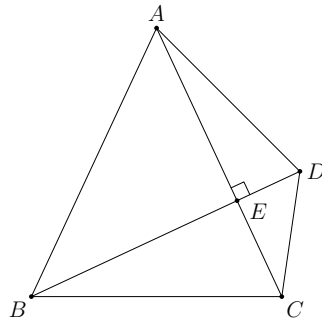
10. (2010 AMC 10A, Problem #14) Triangle ABC has $AB = 2 \cdot AC$. Let D and E be on $\overline{AB}$ and $\overline{BC}$, respectively, such that $\angle BAE = \angle ACD$. Let F be the intersection of segments AE and CD , and suppose that $\triangle CFE$ is equilateral. What is $\angle ACB$?
- (A) 60° (B) 75° (C) 90° (D) 105° (E) 120°

Advanced.

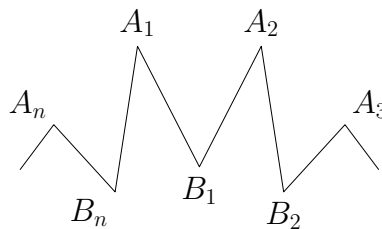
11. (1994 AHSME, Problem #26) A regular polygon of m sides is exactly enclosed (no overlaps, no gaps) by m regular polygons of n sides each. (Shown here for $m = 4$, $n = 8$.) If $m = 10$, what is the value of n ?



- (A) 5 (B) 6 (C) 14 (D) 20 (E) 26
12. (1996 AHSME, Problem #21) Triangles ABC and ABD are isosceles with $AB = AC = BD$, and $\overline{BD}$ intersects $\overline{AC}$ at E . If $\overline{BD} \perp \overline{AC}$, then $\angle C + \angle D$ is



- (A) 115° (B) 120° (C) 130° (D) 135° (E) not uniquely determined
13. (1992 AHSME, Problem #20) Part of an “ n -pointed regular star” is shown. It is a simple closed polygon in which all $2n$ edges are congruent, angles $A_1, A_2, \dots, A_n$ are congruent, and angles $B_1, B_2, \dots, B_n$ are congruent. If the acute angle at A_1 is 10° less than the acute angle at B_1 , then $n =$



- (A) 12 (B) 18 (C) 24 (D) 36 (E) 60
14. (1999 AHSME, Problem #26) Three non-overlapping regular plane polygons, at least two of which are congruent, all have sides of length 1. The polygons meet at a point A in such a way that the sum of the three interior angles at A is 360° . Thus the three polygons form a new polygon with A as an interior point. What is the largest possible perimeter that this polygon can have?
- (A) 12 (B) 14 (C) 18 (D) 21 (E) 24