

ANGLE CHASING AND POLYGONS FOR AMC 10: SOLUTIONS

ANDY JO

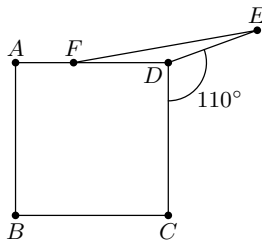
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1. SOLUTIONS

Beginner.

1. (2021 Fall AMC 10A, Problem #7) As shown in the figure below, point E lies in the opposite half-plane determined by line CD from point A so that $\angle CDE = 110^\circ$. Point F lies on $\overline{AD}$ so that $DE = DF$, and $ABCD$ is a square. What is the degree measure of $\angle AFE$?



- (A) 160 (B) 164 (C) 166 (D) 170 (E) 174

Strategy. Triangle DEF is isosceles. Find its vertex angle at D from the angles around D .

Solution. The angles around the point D add to 360° . Two of them are $\angle ADC = 90^\circ$, an angle of the square, and $\angle CDE = 110^\circ$; they do not overlap, because E and A are on opposite sides of line CD . The third angle is the one between DE and DF , since F lies on DA :

$$\angle FDE = 360^\circ - 90^\circ - 110^\circ = 160^\circ.$$

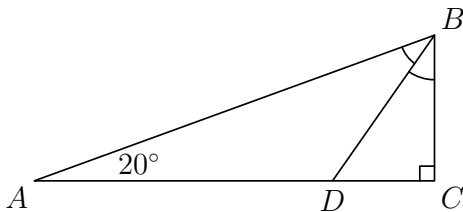
Since $DE = DF$, triangle DEF is isosceles with vertex angle 160° , so

$$\angle DFE = \frac{180^\circ - 160^\circ}{2} = 10^\circ.$$

The points A , F , D lie on a line, so $\angle AFE = 180^\circ - \angle DFE = 170^\circ$.

Answer: (D) 170. □

2. (1986 AHSME, Problem #3) In the figure, $\triangle ABC$ has a right angle at C and $\angle A = 20^\circ$. If BD is the bisector of $\angle ABC$, then $\angle BDC =$



- (A) 40° (B) 45° (C) 50° (D) 55° (E) 60°

Strategy. Find $\angle ABC$, take half of it, and use the right triangle BCD .

Solution. The acute angles of the right triangle ABC add to 90° , so $\angle ABC = 90^\circ - 20^\circ = 70^\circ$. The bisector BD gives $\angle DBC = 35^\circ$. In the right triangle BCD , the acute angles also add to 90° , so

$$\angle BDC = 90^\circ - 35^\circ = 55^\circ.$$

Equivalently, $\angle BDC$ is an exterior angle of triangle ABD , so $\angle BDC = \angle A + \angle ABD = 20^\circ + 35^\circ = 55^\circ$.

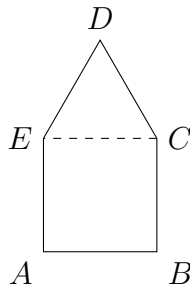
Answer: (D) 55° . □

3. (2007 AMC 10B, Problem #7) All sides of the convex pentagon $ABCDE$ are of equal length, and $\angle A = \angle B = 90^\circ$. What is the degree measure of $\angle E$?

(A) 90 (B) 108 (C) 120 (D) 144 (E) 150

Strategy. The two right angles make $ABCE$ a square. Then triangle CDE has three equal sides.

Solution. Draw the segment CE (dashed in the figure).

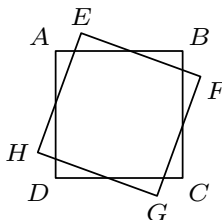


Since $\angle A = \angle B = 90^\circ$, the sides AE and BC are both perpendicular to AB . They lie on the same side of AB , because the pentagon is convex, and they have the same length. So $ABCE$ is a parallelogram with a right angle, that is, a rectangle, and since $AB = BC$ it is a square. Hence $CE = AB$, and triangle CDE has $CD = DE = EC$. It is equilateral, so $\angle CED = 60^\circ$. The pentagon is convex, so D lies outside the square, and

$$\angle E = \angle AEC + \angle CED = 90^\circ + 60^\circ = 150^\circ.$$

Answer: (E) 150. □

4. (2023 AMC 10B, Problem #7) Square $ABCD$ is rotated 20° clockwise about its center to obtain square $EFGH$, as shown below. What is the degree measure of $\angle EAB$?



(A) 24° (B) 35° (C) 30° (D) 32° (E) 20°

Strategy. Use the center O of the square. The rotation moves A to E without changing its distance from O , so triangle AOE is isosceles.

Solution. Let O be the center of square $ABCD$. The rotation about O takes A to E , so $OE = OA$ and $\angle AOE = 20^\circ$. Triangle AOE is isosceles with vertex angle 20° , so

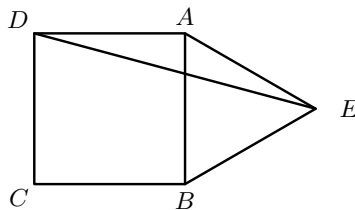
$$\angle OAE = \frac{180^\circ - 20^\circ}{2} = 80^\circ.$$

The diagonal AC of the square passes through O and bisects the right angle at A , so $\angle OAB = 45^\circ$. As the figure shows, ray AB lies between rays AO and AE , so

$$\angle EAB = \angle OAE - \angle OAB = 80^\circ - 45^\circ = 35^\circ.$$

Answer: (B) 35° . □

5. (1979 AHSME, Problem #3) In the adjoining figure, $ABCD$ is a square, ABE is an equilateral triangle and point E is outside square $ABCD$. What is the measure of $\angle AED$ in degrees?



- (A) 10 (B) 12.5 (C) 15 (D) 20 (E) 25

Strategy. The square and the triangle share side AB , so $AD = AB = AE$, and triangle ADE is isosceles.

Solution. Since $ABCD$ is a square, $AD = AB$, and since ABE is equilateral, $AE = AB$. So triangle DAE is isosceles with $AD = AE$. Because E is outside the square, its vertex angle is

$$\angle DAE = \angle DAB + \angle BAE = 90^\circ + 60^\circ = 150^\circ,$$

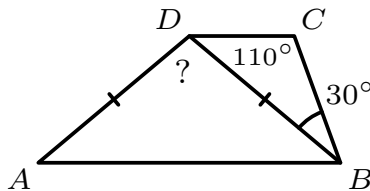
and its base angles are

$$\angle AED = \angle ADE = \frac{180^\circ - 150^\circ}{2} = 15^\circ.$$

Answer: (C) 15. □

Intermediate.

6. (1981 AHSME, Problem #5) In trapezoid $ABCD$, sides AB and CD are parallel, and diagonal BD and side AD have equal length. If $\angle DCB = 110^\circ$ and $\angle CBD = 30^\circ$, then $\angle ADB =$



- (A) 80° (B) 90° (C) 100° (D) 110° (E) 120°

Strategy. The diagonal BD is a transversal of the parallel sides. It carries an angle of triangle BCD over to the isosceles triangle ABD .

Solution. In triangle BCD ,

$$\angle BDC = 180^\circ - 110^\circ - 30^\circ = 40^\circ.$$

Since $AB \parallel CD$, the alternate interior angles $\angle ABD$ and $\angle BDC$ are equal, so $\angle ABD = 40^\circ$. Since $AD = BD$, triangle ABD is isosceles with $\angle DAB = \angle DBA = 40^\circ$. Therefore

$$\angle ADB = 180^\circ - 40^\circ - 40^\circ = 100^\circ.$$

Answer: (C) 100° . □

7. (1960 AHSME, Problem #27) Let S be the sum of the interior angles of a polygon P for which each interior angle is $7\frac{1}{2}$ times the exterior angle at the same vertex. Then:
- (A) $S = 2660^\circ$ and P may be regular (B) $S = 2660^\circ$ and P is not regular
 (C) $S = 2700^\circ$ and P is regular (D) $S = 2700^\circ$ and P is not regular
 (E) $S = 2700^\circ$ and P may or may not be regular

Strategy. At each vertex the interior and exterior angles add to 180° , which fixes the exterior angle. The exterior angles add to 360° , which fixes the number of sides.

Solution. Let e be the exterior angle at a vertex. Then the interior angle is $\frac{15}{2}e$, and

$$\frac{15}{2}e + e = 180^\circ, \quad e = \frac{360^\circ}{17}.$$

This holds at every vertex, so all the exterior angles equal $\frac{360^\circ}{17}$. They add to 360° , so the polygon has 17 vertices, and

$$S = (17 - 2) \cdot 180^\circ = 2700^\circ.$$

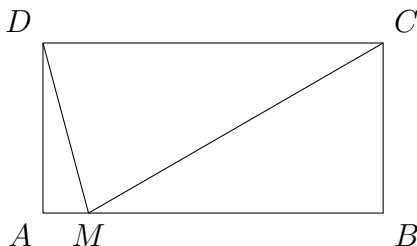
All the angles of P are equal, but nothing is said about its sides. A regular 17-gon has this property. So does a non-regular one: move one side of a regular 17-gon outward, parallel to itself; every angle stays the same, but the sides no longer all have the same length. Hence P may or may not be regular.

Answer: (E) $S = 2700^\circ$ and P may or may not be regular. □

8. (2011 AMC 10B, Problem #18) Rectangle $ABCD$ has $AB = 6$ and $BC = 3$. Point M is chosen on side AB so that $\angle AMD = \angle CMD$. What is the degree measure of $\angle AMD$?
- (A) 15 (B) 30 (C) 45 (D) 60 (E) 75

Strategy. The segment MD bisects $\angle AMC$, and $AB \parallel CD$. A bisector together with a parallel line gives an isosceles triangle.

Solution. The figure shows the rectangle with the segments MD and MC .



Since $AB \parallel CD$, the alternate interior angles $\angle AMD$ and $\angle MDC$ are equal. Together with $\angle AMD = \angle CMD$, this gives $\angle MDC = \angle CMD$, so triangle CDM is isosceles with $CM = CD = 6$.

In the right triangle MBC , the leg $BC = 3$ is half the hypotenuse $CM = 6$. By the theorem on half of an equilateral triangle, the angle opposite BC is $\angle CMB = 30^\circ$. Hence $\angle AMC = 180^\circ - 30^\circ = 150^\circ$, and since MD bisects $\angle AMC$,

$$\angle AMD = \frac{150^\circ}{2} = 75^\circ.$$

Answer: (E) 75. □

9. (1952 AHSME, Problem #21) The sides of a regular polygon of n sides, $n > 4$, are extended to form a star. The number of degrees at each point of the star is:

(A) $\frac{360}{n}$ (B) $\frac{(n-4)180}{n}$ (C) $\frac{(n-2)180}{n}$ (D) $180 - \frac{90}{n}$ (E) $\frac{180}{n}$

Strategy. Each point of the star is the top vertex of a triangle whose base is a side of the polygon. Its base angles are exterior angles of the polygon.

Solution. Each exterior angle of a regular n -gon is $\frac{360}{n}$ degrees. Take one side of the polygon and extend the two sides next to it beyond it. They make the exterior angles of the polygon with this side, so together with it they form a triangle with two base angles of $\frac{360}{n}$ degrees each. (Since $n > 4$, these two angles add to less than 180° , so the extended sides do meet.) The point angle of the star is the third angle of this triangle:

$$180 - 2 \cdot \frac{360}{n} = 180 - \frac{720}{n} = \frac{(n-4)180}{n}.$$

For example, $n = 5$ gives the 36° points of the regular five-pointed star.

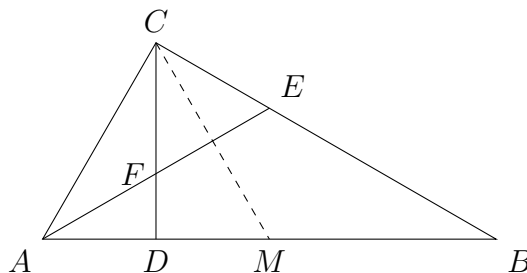
Answer: (B) $\frac{(n-4)180}{n}$. □

10. (2010 AMC 10A, Problem #14) Triangle ABC has $AB = 2 \cdot AC$. Let D and E be on $\overline{AB}$ and $\overline{BC}$, respectively, such that $\angle BAE = \angle ACD$. Let F be the intersection of segments AE and CD , and suppose that $\triangle CFE$ is equilateral. What is $\angle ACB$?

(A) 60° (B) 75° (C) 90° (D) 105° (E) 120°

Strategy. The equilateral triangle gives $\angle AFC = 120^\circ$, and this forces $\angle BAC = 60^\circ$. Then use $AB = 2 \cdot AC$ with the midpoint of AB .

Solution. The figure shows the triangle, with the midpoint M of AB used below.



Let $x = \angle BAE = \angle ACD$. Since A, F, E lie on a line and $\angle CFE = 60^\circ$, the angle $\angle AFC$ is $180^\circ - 60^\circ = 120^\circ$. In triangle AFC the other two angles add to 60° , and $\angle FCA = \angle DCA = x$, so $\angle FAC = 60^\circ - x$. Therefore

$$\angle BAC = \angle BAE + \angle EAC = x + (60^\circ - x) = 60^\circ.$$

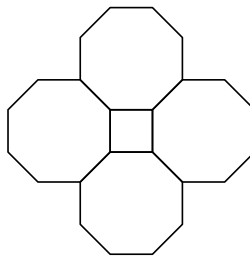
Now let M be the midpoint of AB . Then $AM = \frac{1}{2}AB = AC$, so triangle AMC is isosceles with a 60° angle at A , which makes it equilateral: $CM = AM = MB$ and $\angle ACM = \angle AMC = 60^\circ$. Triangle CMB is isosceles with $CM = MB$ and vertex angle $\angle CMB = 180^\circ - 60^\circ = 120^\circ$, so $\angle MCB = 30^\circ$. Hence

$$\angle ACB = \angle ACM + \angle MCB = 60^\circ + 30^\circ = 90^\circ.$$

Answer: (C) 90° . □

Advanced.

11. (1994 AHSME, Problem #26) A regular polygon of m sides is exactly enclosed (no overlaps, no gaps) by m regular polygons of n sides each. (Shown here for $m = 4$, $n = 8$.) If $m = 10$, what is the value of n ?



- (A) 5 (B) 6 (C) 14 (D) 20 (E) 26

Strategy. Look at one vertex of the 10-gon. Its angle and the angles of two of the n -gons fill the full 360° around that vertex.

Solution. As in the figure, each of the n -gons is attached along one side of the m -gon, and the two n -gons on neighboring sides of the m -gon meet along a common side. So at each vertex of the m -gon, three angles fit around the point with no gaps or overlaps: one angle of the m -gon and one angle of each of two n -gons. (For $m = 4$ and $n = 8$: $90^\circ + 135^\circ + 135^\circ = 360^\circ$.)

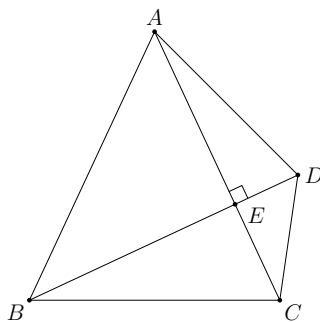
Each interior angle of a regular 10-gon is $180^\circ - 36^\circ = 144^\circ$. If a is the interior angle of the n -gon, then

$$144^\circ + 2a = 360^\circ, \quad a = 108^\circ.$$

So the exterior angle of the n -gon is $180^\circ - 108^\circ = 72^\circ = \frac{360^\circ}{n}$, and $n = 5$.

Answer: (A) 5. □

12. (1996 AHSME, Problem #21) Triangles ABC and ABD are isosceles with $AB = AC = BD$, and $\overline{BD}$ intersects $\overline{AC}$ at E . If $\overline{BD} \perp \overline{AC}$, then $\angle C + \angle D$ is



- (A) 115° (B) 120° (C) 130° (D) 135° (E) not uniquely determined

Strategy. The figure is not determined by the given facts, so let $\alpha = \angle BAC$. Express $\angle C$ and $\angle D$ in terms of α using the two isosceles triangles and the right angle at E .

Solution. Let $\alpha = \angle BAC$. Triangle ABC is isosceles with $AB = AC$ and vertex angle α , so

$$\angle C = 90^\circ - \frac{\alpha}{2}.$$

In the right triangle ABE , the acute angles add to 90° , so $\angle ABE = 90^\circ - \alpha$. Since E lies on BD , this is also $\angle ABD$. Triangle ABD is isosceles with $BA = BD$ and vertex angle

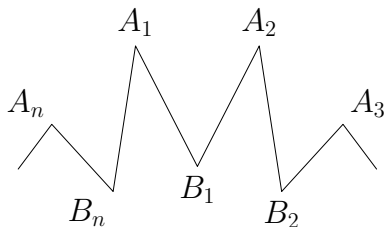
$90^\circ - \alpha$ at B , so its base angle at D is

$$\angle D = \frac{180^\circ - (90^\circ - \alpha)}{2} = 45^\circ + \frac{\alpha}{2}.$$

Adding, $\angle C + \angle D = 135^\circ$. The angle α cancels, so the sum is the same for every such figure.

Answer: (D) 135° . □

13. (1992 AHSME, Problem #20) Part of an “ n -pointed regular star” is shown. It is a simple closed polygon in which all $2n$ edges are congruent, angles $A_1, A_2, \dots, A_n$ are congruent, and angles $B_1, B_2, \dots, B_n$ are congruent. If the acute angle at A_1 is 10° less than the acute angle at B_1 , then $n =$



- (A) 12 (B) 18 (C) 24 (D) 36 (E) 60

Strategy. The star is a polygon with $2n$ vertices whose sides do not cross, so its interior angles add to $(2n - 2) \cdot 180^\circ$. At the points A_i the interior angle is acute; at the vertices B_i it is a reflex angle.

Solution. Let a be the acute angle at each A_i and b the acute angle at each B_i , so $b = a + 10^\circ$. The star is the region below the zigzag in the figure, so its interior angle at each A_i is a , while at each B_i the acute angle lies outside the star and the interior angle is the reflex angle $360^\circ - b$. The star has $2n$ vertices and its sides do not cross, so its interior angles add to $(2n - 2) \cdot 180^\circ$:

$$na + n(360^\circ - b) = (2n - 2) \cdot 180^\circ = 360^\circ n - 360^\circ.$$

This simplifies to $n(b - a) = 360^\circ$. Since $b - a = 10^\circ$, this gives $n = 36$.

Answer: (D) 36. □

14. (1999 AHSME, Problem #26) Three non-overlapping regular plane polygons, at least two of which are congruent, all have sides of length 1. The polygons meet at a point A in such a way that the sum of the three interior angles at A is 360° . Thus the three polygons form a new polygon with A as an interior point. What is the largest possible perimeter that this polygon can have?

- (A) 12 (B) 14 (C) 18 (D) 21 (E) 24

Strategy. Write the angle condition with the formula for the interior angle of a regular polygon, find every possible set of three polygons, and count the sides on the outside.

Solution. Let the two congruent polygons have n sides and the third have m sides. The angle condition is

$$2 \left(180 - \frac{360}{n} \right) + \left(180 - \frac{360}{m} \right) = 360,$$

which simplifies to $\frac{720}{n} + \frac{360}{m} = 180$, or $\frac{4}{n} + \frac{2}{m} = 1$. Solving for n ,

$$n = \frac{4m}{m - 2} = 4 + \frac{8}{m - 2},$$

so $m - 2$ divides 8. Hence $m \in \{3, 4, 6, 10\}$, which gives

$$(n, m) = (12, 3), (8, 4), (6, 6), (5, 10).$$

Around A , each pair of the three polygons shares one side that ends at A . These three shared sides lie inside the new polygon, and every other side is on its boundary. So the perimeter is $(2n + m) - 2 \cdot 3$:

$$24 + 3 - 6 = 21, \quad 16 + 4 - 6 = 14, \quad 12 + 6 - 6 = 12, \quad 10 + 10 - 6 = 14.$$

The largest is 21, from two regular 12-gons and an equilateral triangle ($150^\circ + 150^\circ + 60^\circ = 360^\circ$).

Answer: (D) 21.

□

2. ANSWER KEY

Problem	Letter	Value
1	D	170
2	D	55°
3	E	150
4	B	35°
5	C	15
6	C	100°
7	E	$S = 2700^\circ$ and P may or may not be regular
8	E	
9	B	$\frac{(n-4)180}{n}$
10	C	90°
11	A	5
12	D	135°
13	D	36
14	D	21