ANGLE CHASING AND POLYGONS FOR AMC 10: PRACTICE SET SOLUTIONS

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1. Solutions

Beginner.

1. (2011 AMC 10B, Problem #7) The sum of two angles of a triangle is $\frac{6}{5}$ of a right angle, and one of these two angles is 30° larger than the other. What is the degree measure of the largest angle in the triangle?

(A) 69 **(B)** 72 **(C)** 90 **(D)** 102 **(E)** 108

Strategy. The triangle angle sum gives the third angle directly from the sum of the other two. To see which angle is the largest, find the two given angles as well.

Solution. Six fifths of a right angle is $\frac{6}{5} \cdot 90^\circ = 108^\circ$. If the smaller of the two angles is x, the larger one is $x + 30^\circ$, and

$$x + (x + 30^\circ) = 108^\circ,$$

so $x = 39^\circ$. The two angles are 39° and 69°. By the triangle angle sum, the third angle is

$$180^\circ - 108^\circ = 72^\circ,$$

which is larger than both of them. So the largest angle of the triangle measures 72°.

Answer: (B) 72. □

2. (2007 AMC 10A, Problem #8) Triangles ABC and ADC are isosceles with $AB = BC$ and $AD = DC$. Point D is inside $\triangle ABC$, $\angle ABC = 40^\circ$, and $\angle ADC = 140^\circ$. What is the degree measure of $\angle BAD$?

(A) 20 **(B)** 30 **(C)** 40 **(D)** 50 **(E)** 60

Strategy. Both triangles have base AC. Find the base angle at A of each triangle from its vertex angle; $\angle BAD$ is the difference of the two.

Solution. Triangle ABC is isosceles with $BA = BC$, so its base is AC and its vertex angle is $\angle ABC = 40^\circ$. Its base angle at A is

$$\angle BAC = \frac{180^\circ - 40^\circ}{2} = 70^\circ.$$

Triangle ADC is isosceles with $DA = DC$ and vertex angle $\angle ADC = 140^\circ$, so its base angle at A is

$$\angle DAC = \frac{180^\circ - 140^\circ}{2} = 20^\circ.$$

Since D is inside triangle ABC, the ray AD lies between the rays AB and AC. Therefore

$$\angle BAD = \angle BAC - \angle DAC = 70^\circ - 20^\circ = 50^\circ.$$

Answer: (D) 50. □

3. (2015 AMC 12B, Problem #7) A regular 15-gon has L lines of symmetry, and the smallest positive angle for which it has rotational symmetry is R degrees. What is $L + R$?

(A) 24 **(B)** 27 **(C)** 32 **(D)** 39 **(E)** 54

Strategy. Every symmetry of a regular polygon keeps its center O in place. Count the lines of symmetry through O, and find the smallest turn about O that carries the vertices onto the vertices.

Solution. Let O be the center of the regular 15-gon, the only point at the same distance from all of its vertices. A reflection or a rotation that maps the polygon onto itself carries the vertices onto the vertices, so it keeps O in place. Hence every line of symmetry passes through O, and every rotational symmetry is a turn about O. The vertices are equally spaced around O, and the central angle between neighboring vertices is $\frac{360^\circ}{15} = 24^\circ$.

The reflection in the line through O and a vertex V swaps the two neighbors of V, then the next two vertices, and so on, so this line is a line of symmetry. Since 15 is odd, the line leaves the polygon through the midpoint of the opposite side, so the 15 vertices give 15 different lines. There are no other lines of symmetry: the reflection in such a line pairs off the vertices that are not on the line, and since 15 is odd, at least one vertex lies on it. Hence $L = 15$.

A turn of 24° about O moves each vertex to the next one, so it is a rotational symmetry. A turn through a positive angle less than 24° moves each vertex to a point strictly between two neighboring vertices, so it is not. Hence $R = 24$, and

$$L + R = 15 + 24 = 39.$$

Answer: (D) 39. □

4. (1962 AHSME, Problem #20) The angles of a pentagon are in arithmetic progression. One of the angles, in degrees, must be:

(A) 108 **(B)** 90 **(C)** 72 **(D)** 54 **(E)** 36

Strategy. The five angles add to 540°. In an arithmetic progression with an odd number of terms, the middle term is the average of all the terms.

Solution. The interior angles of a pentagon add to $(5-2)\cdot 180^\circ = 540^\circ$. Write the five angles in increasing order as

$$a-2d, \quad a-d, \quad a, \quad a+d, \quad a+2d,$$

where a is the middle angle and $d \geq 0$ is the common difference. Their sum is $5a$, so $5a = 540^\circ$ and $a = 108^\circ$. Thus every such pentagon has an angle of 108°, whatever d is. None of the other values is forced: the angles of a regular pentagon form an arithmetic progression with $d = 0$, and all five of them are 108°.

Answer: (A) 108. □

5. (1968 AHSME, Problem #6) Let side AD of convex quadrilateral $ABCD$ be extended through D, and let side BC be extended through C, to meet in point E. Let S represent the degree-sum of angles CDE and DCE, and let S' represent the degree-sum of angles BAD and ABC. If $r = \frac{S}{S'}$, then:

(A) $r = 1$ sometimes, $r > 1$ sometimes **(B)** $r = 1$ sometimes, $r < 1$ sometimes
(C) $0 < r < 1$ **(D)** $r > 1$ **(E)** $r = 1$

Strategy. Extending the sides AD and BC beyond side CD makes a triangle on CD whose base angles are the exterior angles of the quadrilateral at D and C. Compare these with the angles of $ABCD$, which add to 360°.

Solution. Since E lies on the extension of AD beyond D, the angle $\angle CDE$ lies between side DC and the extension of side AD. It is the exterior angle of $ABCD$ at D, so $\angle CDE = 180^\circ - \angle ADC$. In the same way, $\angle DCE$ is the exterior angle at C, and $\angle DCE = 180^\circ - \angle BCD$. Hence

$$S = 360^\circ - \angle ADC - \angle BCD.$$

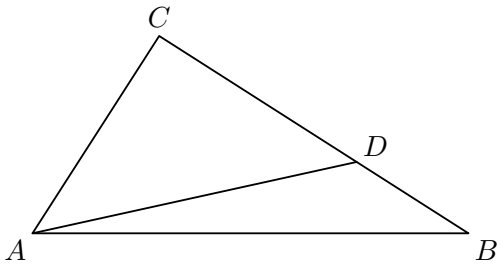
The four angles of the convex quadrilateral $ABCD$ add to $360°$, so $360° - \angle ADC - \angle BCD = \angle BAD + \angle ABC = S'$. Therefore $S = S'$, and $r = 1$ for every such quadrilateral.

The angle sums of the triangles CDE and ABE, which share the angle at E, give the same result: $S = 180° - \angle E = S'$.

Answer: (E) $r = 1$. □

Intermediate.

6. (1957 AHSME, Problem #44) In triangle ABC, $AC = CD$ and $\angle CAB - \angle ABC = 30°$. Then $\angle BAD$ is:



(A) $30°$ **(B)** $20°$ **(C)** $22\frac{1}{2}°$ **(D)** $10°$ **(E)** $15°$

Strategy. Let $x = \angle BAD$. Find $\angle CAD$ in two steps: first the exterior angle of triangle ABD at D, then the base angles of the isosceles triangle ACD.

Solution. In the figure, D lies on side BC. Let $x = \angle BAD$ and $b = \angle ABC$. The angle $\angle ADC$ is an exterior angle of triangle ABD, so by the exterior angle theorem

$$\angle ADC = \angle ABD + \angle BAD = b + x.$$

Since $CA = CD$, triangle ACD is isosceles with base AD, and its base angles are equal: $\angle CAD = \angle CDA = b + x$. The ray AD lies between the rays AB and AC, so

$$\angle CAB = \angle CAD + \angle DAB = (b + x) + x = b + 2x.$$

The condition $\angle CAB - \angle ABC = 30°$ now reads $2x = 30°$, so $x = 15°$. The angle b drops out, so every triangle with these properties gives the same answer.

Answer: (E) $15°$. □

7. (1961 AHSME, Problem #27) Given two equiangular polygons P_1 and P_2 with different numbers of sides; each angle of P_1 is x degrees and each angle of P_2 is kx degrees, where k is an integer greater than 1. The number of possibilities for the pair (x, k) is:

(A) infinite **(B)** finite, but greater than two **(C)** two **(D)** one **(E)** zero

Strategy. Each angle of an equiangular n-gon is $180° - \frac{360°}{n}$, so it is at least $60°$ and less than $180°$. These two bounds leave very few choices for x and k.

Solution. In an equiangular n-gon the n exterior angles are equal and add to $360°$, so each interior angle is $180° - \frac{360°}{n}$. Since $n \geq 3$, every such angle satisfies

$$60° \leq 180° - \frac{360°}{n} < 180°.$$

The angle kx of P_2 is less than 180 and $k \geq 2$, so $2x \leq kx < 180$ and $x < 90$. An angle $180 - \frac{360}{n}$ is less than 90 only when $\frac{360}{n} > 90$, that is, when $n = 3$. So P_1 is a triangle and $x = 60$. Then $60k < 180$ gives $k < 3$, so $k = 2$, and P_2 has angles of $120°$, as a

regular hexagon does. A triangle and a hexagon have different numbers of sides, so the pair $(x, k) = (60, 2)$ does occur, and it is the only one.

Answer: (D) one. □

8. (1991 AHSME, Problem #12) The measures (in degrees) of the interior angles of a convex hexagon form an arithmetic sequence of positive integers. Let m° be the measure of the largest interior angle of the hexagon. The largest possible value of m° is

(A) 165° **(B)** 167° **(C)** 170° **(D)** 175° **(E)** 179°

Strategy. The largest interior angle goes with the smallest exterior angle. The six exterior angles also form an arithmetic sequence, they add to 360°, and each of them is positive.

Solution. Each exterior angle is 180° minus the interior angle at the same vertex, so the exterior angles also form an arithmetic sequence of integers (in degrees), with the same common difference $d \geq 0$. Write them in increasing order as e, $e + d$, ..., $e + 5d$. The smallest exterior angle e belongs to the largest interior angle, so $m = 180 - e$. The exterior angles add to 360°:

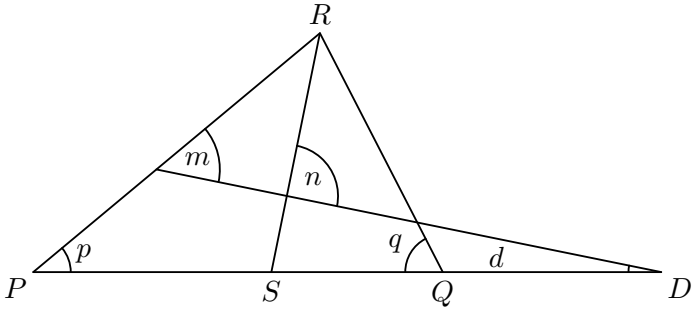
$$6e + 15d = 360, \qquad \text{so} \qquad e = 60 - \frac{5d}{2}.$$

Since e is an integer, d is even. Since the hexagon is convex, $e > 0$, so $5d < 120$ and $d < 24$. The largest even value is $d = 22$, which gives $e = 5$. Hence $m = 180 - e \leq 175$.

The value 175 is attained. For $d = 22$ the interior angles are 65°, 87°, 109°, 131°, 153° and 175°, and a convex hexagon with these angles exists. Cut off each corner of an equilateral triangle by a short segment. At each corner, the segment and the two sides form a small triangle with angles 60°, u and v, where $u + v = 120^\circ$, and the hexagon gets the angles $180^\circ - u$ and $180^\circ - v$ at the ends of the segment; these two angles add to 240°. The pairs 65° and 175°, 87° and 153°, and 109° and 131° each add to 240°, so the cuts with $(u, v) = (115^\circ, 5^\circ)$, $(93^\circ, 27^\circ)$ and $(71^\circ, 49^\circ)$ give a hexagon with exactly these six angles.

Answer: (D) 175°. □

9. (1954 AHSME, Problem #37) Given triangle PQR with $\overline{RS}$ bisecting $\angle R$, $\overline{PQ}$ extended to D, and $\angle n$ a right angle, then:



(A) $\angle m = \frac{1}{2}(\angle p - \angle q)$ **(B)** $\angle m = \frac{1}{2}(\angle p + \angle q)$ **(C)** $\angle d = \frac{1}{2}(\angle q + \angle p)$

(D) $\angle d = \frac{1}{2}\angle m$ **(E)** none of these is correct

Strategy. Since $\angle n$ is a right angle, $\angle m$ and half of $\angle R$ are the two acute angles of a right triangle. Write half of $\angle R$ in terms of $\angle p$ and $\angle q$.

Solution. In the figure, the line through D meets side PR at a point E and the bisector RS at a point F, so that $\angle m = \angle REF$ and $\angle n = \angle RFD = 90^\circ$. The angles $\angle RFE$

and $\angle n$ form a straight line, so $\angle RFE = 180^\circ - \angle n = 90^\circ$. By the triangle angle sum in triangle PQR, $\angle R = 180^\circ - \angle p - \angle q$. Since RS bisects $\angle R$,

$$\angle ERF = \frac{180^\circ - \angle p - \angle q}{2} = 90^\circ - \frac{\angle p + \angle q}{2}.$$

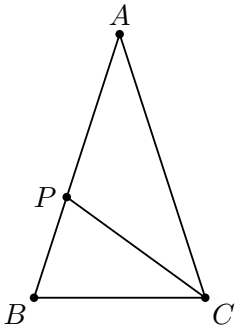
In the right triangle REF, the two acute angles add to 90°, so

$$\angle m = 90^\circ - \angle ERF = \frac{\angle p + \angle q}{2}.$$

So $\angle m$ is half of the exterior angle of triangle PQR at R, which equals $\angle p + \angle q$ by the exterior angle theorem. The other choices fail in general: $\angle m$ is an exterior angle of triangle PED, so $\angle m = \angle p + \angle d$ and $\angle d = \frac{1}{2}(\angle q - \angle p)$.

Answer: (B) $\angle m = \dfrac{1}{2}(\angle p + \angle q)$. □

10. (1994 AHSME, Problem #13) In triangle ABC, $AB = AC$. If there is a point P strictly between A and B such that $AP = PC = CB$, then $\angle A =$



(A) 30° **(B)** 36° **(C)** 48° **(D)** 60° **(E)** 72°

Strategy. The equal segments $AP = PC = CB$ form an isosceles chain. Call $\angle A = x$, walk along the chain with the exterior angle theorem, and finish with the base angles of triangle ABC.

Solution. Let $x = \angle A$.

- Since $PA = PC$, triangle APC is isosceles and $\angle PCA = \angle PAC = x$.
- The angle $\angle BPC$ is an exterior angle of triangle APC, so $\angle BPC = x + x = 2x$.
- Since $CP = CB$, triangle CPB is isosceles with base PB, so $\angle CBP = \angle CPB = 2x$. Thus $\angle ABC = 2x$.

Since $AB = AC$, triangle ABC has equal base angles, $\angle ACB = \angle ABC = 2x$. The angle sum in triangle ABC gives

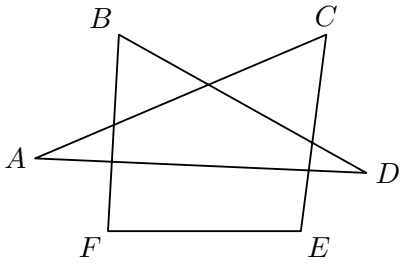
$$x + 2x + 2x = 180^\circ,$$

so $x = 36^\circ$.

Answer: (B) 36°. □

Advanced.

11. (1972 AHSME, Problem #21) If the sum of the measures in degrees of angles A, B, C, D, E, and F in the figure is $90n$, then n is equal to:



(A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

Strategy. Argue as in the proof that the point angles of a five-pointed star add to 180°. Two triangles cut off by AD carry the angles at A, B, C and D, in pairs, to two angles of a quadrilateral whose other two angles are the angles at E and F.

Solution. Let AD cross BF at X and CE at Y, so that A, X, Y, D lie on AD in this order, as in the figure.

- The angles of triangle BXD at B and D are the angles B and D of the figure. Since F lies on BX extended beyond X, and Y lies on XD, the angle $\angle FXY$ is an exterior angle of this triangle, so $\angle FXY = \angle B + \angle D$.
- The angles of triangle ACY at A and C are the angles A and C of the figure. Since E lies on CY extended beyond Y, and X lies on YA, the angle $\angle XYE$ is an exterior angle of this triangle, so $\angle XYE = \angle A + \angle C$.

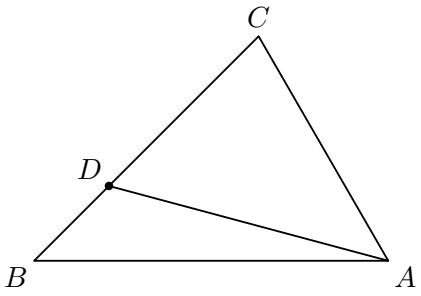
The angles of the quadrilateral $EFXY$ at E and F are the angles E and F of the figure, and its four angles add to 360°. Therefore

$$(\angle A + \angle C) + (\angle B + \angle D) + \angle E + \angle F = \angle XYE + \angle FXY + \angle E + \angle F = 360°.$$

Since $360 = 90 \cdot 4$, the sum is $90n$ with $n = 4$.

Answer: (C) 4. □

12. (2001 AMC 12, Problem #24) In triangle ABC, $\angle ABC = 45°$. Point D is on $\overline{BC}$ so that $2 \cdot BD = CD$ and $\angle DAB = 15°$. Find $\angle ACB$.



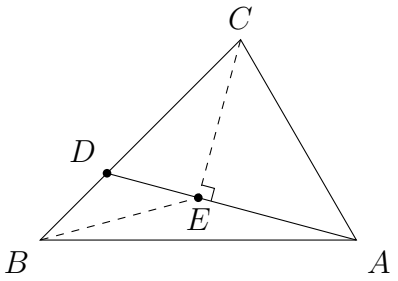
(A) 54° **(B)** 60° **(C)** 72° **(D)** 75° **(E)** 90°

Strategy. The exterior angle of triangle ABD at D is 60°. Add the foot E of the perpendicular from C to AD: it makes half of an equilateral triangle with $DE = \frac{1}{2}CD = BD$, and then E turns out to be equally far from A, B and C.

Solution. By the exterior angle theorem in triangle ABD,

$$\angle ADC = \angle ABD + \angle BAD = 45° + 15° = 60°.$$

Drop the perpendicular from C to line AD, and call its foot E. In the figure the segments CE and BE are dashed.



In the right triangle CED, the angle at D is $60°$, so $\angle ECD = 30°$. By the theorem on half of an equilateral triangle, the leg DE opposite this angle is half the hypotenuse CD:

$$DE = \tfrac{1}{2}CD = BD.$$

In triangle ABD, the side DA is opposite the $45°$ angle and the side DB is opposite the smaller $15°$ angle, so $DA > DB = DE$, and E lies between D and A. Now chase the isosceles triangles.

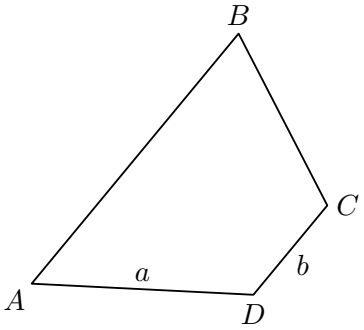
- Triangle BDE has $DB = DE$ and vertex angle $\angle BDE = 180° - \angle ADC = 120°$, so its base angles are $\angle DBE = \angle DEB = 30°$.
- Then $\angle EBA = \angle DBA - \angle DBE = 45° - 30° = 15° = \angle EAB$, so triangle EAB is isosceles with $EA = EB$.
- In triangle EBC, $\angle EBC = \angle DBE = 30°$ and $\angle ECB = \angle ECD = 30°$, so it is isosceles with $EB = EC$.

So $EA = EC$, and AEC is a right triangle with equal legs, whose acute angles are $45°$. Hence $\angle ACE = 45°$ and

$$\angle ACB = \angle ACE + \angle ECB = 45° + 30° = 75°.$$

Answer: (D) $75°$. □

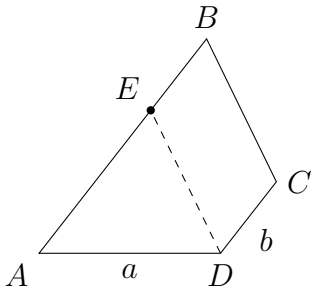
13. (1970 AHSME, Problem #30) In the accompanying figure, segments AB and CD are parallel, the measure of angle D is twice that of angle B, and the measures of segments AD and CD are a and b, respectively. Then the measure of AB is equal to:



(A) $\frac{1}{2}a + 2b$ **(B)** $\frac{3}{2}b + \frac{3}{4}a$ **(C)** $2a - b$ **(D)** $4b - \frac{1}{2}a$ **(E)** $a + b$

Strategy. Draw through D the line parallel to BC. It cuts $ABCD$ into a parallelogram and a triangle, and the angles at the parallel lines show that the triangle is isosceles.

Solution. Let $\angle ABC = \beta$, so that $\angle ADC = 2\beta$. Draw the line through D parallel to BC, and let it meet AB at E. The figure is drawn to scale with $a = 2$, $b = 1$ and $\beta = 64°$, and DE is dashed.



Since $DE \parallel CB$ and $EB \parallel DC$, the quadrilateral $EBCD$ is a parallelogram. Its opposite sides are equal, so $EB = DC = b$, and its opposite angles are equal, so $\angle EDC = \angle EBC = \beta$. Since $\angle EDC = \beta$ is less than $\angle ADC = 2\beta$, the point E lies between A and B, and

$$\angle ADE = \angle ADC - \angle EDC = 2\beta - \beta = \beta.$$

The parallel lines DE and CB are both crossed by AB, so the corresponding angles $\angle AED$ and $\angle ABC$ are equal: $\angle AED = \beta$. Triangle ADE has two angles equal to β, so it is isosceles with $AE = AD = a$. Therefore

$$AB = AE + EB = a + b.$$

Equivalently, DE bisects $\angle ADC$, and since $AB \parallel DC$, a bisector together with a parallel line gives the isosceles triangle ADE at once.

Answer: (E) $a + b$. □

14. (2022 AIME I, Problem #3) In isosceles trapezoid $ABCD$, parallel bases $\overline{AB}$ and $\overline{CD}$ have lengths 500 and 650, respectively, and $AD = BC = 333$. The angle bisectors of $\angle A$ and $\angle D$ meet at P, and the angle bisectors of $\angle B$ and $\angle C$ meet at Q. Find PQ.

Strategy. A bisector together with a parallel line makes an isosceles triangle. Extend the bisector of $\angle A$ to the base CD; the bisector of $\angle D$ then passes through the midpoint of the extended bisector, and that midpoint is P.

Solution. Let the bisector of $\angle A$ meet line CD at X. Since $AB \parallel CD$, the alternate interior angles $\angle AXD$ and $\angle XAB$ are equal, and $\angle XAB = \angle XAD$ because AX bisects $\angle A$. So triangle DAX is isosceles with $DX = DA = 333$. Since $333 < 650$, the point X lies on side CD, at distance 333 from D.

In the isosceles triangle DAX, the bisector of the vertex angle $\angle ADX = \angle ADC$ splits the triangle into two congruent triangles (two sides and the angle between them are equal), so it meets AX at the midpoint of AX. Two different lines meet in at most one point, so this midpoint is the point P where the bisectors of $\angle A$ and $\angle D$ meet. In the same way, the bisector of $\angle B$ meets CD at a point Y with $CY = CB = 333$, and Q is the midpoint of BY.

Place the trapezoid in the coordinate plane with $D = (0, 0)$ and $C = (650, 0)$. The trapezoid is isosceles, so the longer base CD extends $\frac{650-500}{2} = 75$ past AB at each end, and $A = (75, h)$ and $B = (575, h)$, where h is the height. Also $X = (333, 0)$ and $Y = (650 - 333, 0) = (317, 0)$. Hence

$$P = \left(\frac{75 + 333}{2}, \frac{h}{2}\right) = \left(204, \frac{h}{2}\right), \qquad Q = \left(\frac{575 + 317}{2}, \frac{h}{2}\right) = \left(446, \frac{h}{2}\right).$$

The points P and Q are at the same height, so $PQ = 446 - 204 = 242$. The height h is not needed.

Answer: 242. □

15. (2004 AIME I, Problem #8) Define a *regular n-pointed star* to be the union of n line segments $\overline{P_1P_2}$, $\overline{P_2P_3}$, ..., $\overline{P_nP_1}$ such that

- the points P_1, P_2, ..., P_n are coplanar and no three of them are collinear,
- each of the n line segments intersects at least one of the other line segments at a point other than an endpoint,
- all of the angles at P_1, P_2, ..., P_n are congruent,
- all of the n line segments $\overline{P_1P_2}$, $\overline{P_2P_3}$, ..., $\overline{P_nP_1}$ are congruent, and
- the path $P_1P_2 \dots P_nP_1$ turns counterclockwise at an angle of less than 180° at each vertex.

There are no regular 3-pointed, 4-pointed, or 6-pointed stars. All regular 5-pointed stars are similar, but there are two non-similar regular 7-pointed stars. How many non-similar regular 1000-pointed stars are there?

Strategy. A regular n-pointed star is built like a regular star polygon: n points equally spaced around a circle, each joined to the point k places further around. Find the values of k that meet the five conditions, and tell the stars apart by the point angle $180° - \frac{360°k}{n}$.

Solution. Equal segments, equal angles and the same turn at every vertex make the path look the same from each of its vertices. So it is built like a regular star polygon: its vertices are n points equally spaced around a circle, and each segment joins a point to the point k places further counterclockwise, for some k with $1 \le k \le n-1$. At every vertex the path turns counterclockwise through $\frac{360°k}{n}$, the central angle that one segment spans. Now test the conditions.

- The turn is less than 180° exactly when $k < \frac{n}{2}$.
- If n and k have a common factor $g > 1$, the path is back at P_1 after $\frac{n}{g}$ steps, so some of the points coincide, which the first condition rules out. So k is relatively prime to n.
- If $k = 1$, the path is a convex regular polygon, and no two of its segments cross. So $k \ge 2$.
- If $2 \le k < \frac{n}{2}$ and k is relatively prime to n, every condition holds. In particular, each segment crosses the segment that starts one point further counterclockwise, because their endpoints alternate around the circle.

By the formula for the point angle of a regular star polygon, the angle at each vertex is $180° - \frac{360°k}{n}$. Different values of k give different angles, so the stars they give are not similar. Hence the number of non-similar regular n-pointed stars is the number of integers k with $2 \le k < \frac{n}{2}$ that are relatively prime to n. For $n = 3$, 4 and 6 there is no such k, for $n = 5$ there is one ($k = 2$), and for $n = 7$ there are two ($k = 2$ and $k = 3$), as the statement says.

For $n = 1000 = 2^3 \cdot 5^3$, an integer is relatively prime to 1000 exactly when it is divisible by neither 2 nor 5. Among $1, 2, \dots, 1000$ there are 500 multiples of 2, 200 multiples of 5 and 100 multiples of 10, so by inclusion-exclusion $1000 - (500 + 200 - 100) = 400$ of these integers are relatively prime to 1000. The integers k and $1000 - k$ are both relatively prime to 1000 or both not, and 500 is not, so the 400 split evenly: 200 of them are less than 500. Removing $k = 1$ leaves 199.

Answer: 199. □

2. Answer Key

Problem	Letter	Value
1	B	72
2	D	50
3	D	39
4	A	108
5	E	$r = 1$
6	E	15°
7	D	one
8	D	175°
9	B	$\angle m = \frac{1}{2}(\angle p + \angle q)$
10	B	36°
11	C	4
12	D	75°
13	E	$a + b$
14		242
15		199