STARS AND BARS, DISTRIBUTIONS, AND CIRCULAR ARRANGEMENTS FOR AMC 10

ANDY JO

Contents

1. Stars and Bars 2
1.1. Combinations with Repetition 2
1.2. Integer Compositions 2
1.3. Selecting Non-Adjacent Objects 3
2. Arrangements in a Circle 5
2.1. Circular Permutations 5
2.2. Necklaces and Bracelets 6
3. Grouping and Partitioning 7
3.1. Labeled Groups 7
3.2. Unlabeled Groups 7
4. The Principle of Inclusion-Exclusion 8
5. The Hockey-Stick Identity 9
6. Problems 10

1. Stars and Bars

This section counts the ways to distribute identical items into distinct containers when a container may hold any number of items. The method is called **stars and bars**.

1.1. **Combinations with Repetition.** In a **combination with repetition**, k items are chosen from n types of items, each type has an unlimited supply, a type may be chosen more than once, and the order of the choices does not matter. For example, a box of 7 donuts chosen from 4 types (glazed, chocolate, powdered and jelly) is a combination with repetition with $n = 4$ and $k = 7$.

Record a selection as a row of k stars ($\star$) and $n-1$ bars ($|$). The stars before the first bar are the items of the first type, the stars between the first and second bars are the items of the second type, and so on. The box with 3 glazed, 0 chocolate, 2 powdered and 2 jelly donuts is recorded as

$$\star\star\star \mid \mid \star\star \mid \star\star.$$

Theorem 1.1.1. *Let $n \geq 1$ and $k \geq 0$ be integers. The number of ways to choose k items from n types of items, where a type may be chosen more than once and order does not matter, is*

$$\binom{n+k-1}{k} = \binom{n+k-1}{n-1}.$$

Proof. Each selection gives exactly one row of k stars and $n-1$ bars. Each such row gives back exactly one selection: the $n-1$ bars split the stars into n blocks, and the number of stars in the ith block is the number of items of the ith type. So there are as many selections as rows. A row has $n+k-1$ positions and is determined by the k positions that hold stars, so there are $\binom{n+k-1}{k}$ rows. Choosing the $n-1$ positions of the bars instead gives $\binom{n+k-1}{n-1}$, the same number. □

For the donuts, $n = 4$ and $k = 7$, so there are $\binom{10}{7} = 120$ different boxes.

1.2. **Integer Compositions.** An **integer composition** of a positive integer k is a way to write k as an ordered sum of positive integers, called its **parts**. For example, $3 = 1+2$ and $3 = 2+1$ are different compositions of 3. The compositions of k into n parts are the positive integer solutions of

$$x_1 + x_2 + \cdots + x_n = k.$$

If parts equal to 0 are allowed, the count becomes the number of nonnegative integer solutions.

Theorem 1.2.1. *Let $n \geq 1$ and $k \geq 0$ be integers. The number of nonnegative integer solutions of $x_1 + x_2 + \cdots + x_n = k$ is*

$$\binom{n+k-1}{n-1}.$$

Proof. Read x_i as the number of items of the ith type in a selection of k items from n types. Each solution is exactly one such selection, so Theorem 1.1.1 gives the count. □

Theorem 1.2.2. *Let n and k be integers with $1 \leq n \leq k$. The number of positive integer solutions of $x_1 + x_2 + \cdots + x_n = k$ is*

$$\binom{k-1}{n-1}.$$

Proof. Set $y_i = x_i - 1$ for each i. The positive integer solutions of $x_1 + \cdots + x_n = k$ correspond one to one to the nonnegative integer solutions of $y_1 + \cdots + y_n = k - n$. By Theorem 1.2.1, there are $\binom{n+(k-n)-1}{n-1} = \binom{k-1}{n-1}$ of them. □

In terms of items, the substitution places one item in each of the n containers first and then distributes the remaining $k-n$ items freely.

Example.

Problem 1.2.3 (2016 AMC 10A, Problem #20)**.** For some particular value of N, when $(a+b+c+d+1)^N$ is expanded and like terms are combined, the resulting expression contains exactly 1001 terms that include all four variables a, b, c, and d, each to some positive power. What is N?
(A) 9 **(B)** 14 **(C)** 16 **(D)** 17 **(E)** 19

Strategy. A term of the expansion is determined by its list of exponents, so count the exponent lists. The exponents of a, b, c and d must be at least 1, and the exponent of the constant 1 may be 0.

Solution. Expanding $(a+b+c+d+1)^N$ gives a sum of products of N factors, each of which is a, b, c, d or 1. A product with n_1 factors a, n_2 factors b, n_3 factors c, n_4 factors d and n_5 factors 1 equals $a^{n_1}b^{n_2}c^{n_3}d^{n_4}$, where the n_i are nonnegative integers with

$$n_1+n_2+n_3+n_4+n_5=N.$$

Every such list of exponents occurs, all coefficients are positive, and the list is determined by n_1, n_2, n_3, n_4 because $n_5 = N-n_1-n_2-n_3-n_4$. So after like terms are combined there is exactly one term for each list. The terms that contain all four variables correspond to the lists with $n_1, n_2, n_3, n_4 \geq 1$ and $n_5 \geq 0$.

Set $n_i' = n_i - 1$ for $i = 1, 2, 3, 4$. The lists correspond one to one to the nonnegative integer solutions of

$$n_1'+n_2'+n_3'+n_4'+n_5=N-4.$$

By Theorem 1.2.1 with 5 variables and sum $N-4$, the number of terms is

$$\binom{(N-4)+5-1}{5-1}=\binom{N}{4}.$$

So $\binom{N}{4}=1001$, that is,

$$N(N-1)(N-2)(N-3)=24\cdot 1001=24\cdot 7\cdot 11\cdot 13=14\cdot 13\cdot 12\cdot 11.$$

Hence $N=14$ works. For integers $N\geq 4$ the product $N(N-1)(N-2)(N-3)$ increases with N, so no other N works.

Answer: (B) 14. □

1.3. **Selecting Non-Adjacent Objects.** A related count chooses objects from a row so that no two chosen objects are next to each other.

Theorem 1.3.1. *Let $0 \leq k \leq n$. The number of ways to choose k of n objects in a row so that no two chosen objects are adjacent is*

$$\binom{n-k+1}{k}.$$

Proof. Mark each of the $n-k$ unchosen objects with $\times$. These marks leave $n-k+1$ gaps: one before the first mark, one between each two neighboring marks, and one after the last mark.

$$__ \times __ \times __ \cdots \times __$$

In a valid selection, each gap holds at most one chosen object, because two chosen objects in the same gap would be adjacent. Conversely, chosen objects in different gaps are separated by at least one mark. So a valid selection is the same as a choice of k of the $n-k+1$ gaps, and there are $\binom{n-k+1}{k}$ such choices. □

If $2k > n+1$, no such selection exists, and the formula gives 0, because $\binom{m}{k} = 0$ when $k > m$: there is no way to choose k of m objects.

Around a circle, the first and last objects are also neighbors, so Theorem 1.3.1 does not apply directly. One way to count is to split into cases by whether one particular object is chosen. If it is chosen, its two neighbors are not, and the objects that may still be chosen form a row. If it is not chosen, the other objects form a row. Another way is to go around the circle and record, for each chosen object, the number of unchosen objects between it and the next chosen object. No two chosen objects are neighbors exactly when all of these numbers are positive, and they add up to the number of unchosen objects.

Examples.

Problem 1.3.2 (2009 AIME II, Problem #6)**.** Let m be the number of five-element subsets that can be chosen from the set of the first 14 natural numbers so that at least two of the five numbers are consecutive. Find the remainder when m is divided by 1000.

Strategy. Each five-element subset either contains two consecutive numbers or contains no two consecutive numbers. Count all five-element subsets and those with no two consecutive numbers, and subtract.

Solution. The first 14 natural numbers are $1, 2, \ldots, 14$. In this row, two numbers are neighbors exactly when they are consecutive. There are $\binom{14}{5} = 2002$ five-element subsets in all. By Theorem 1.3.1 with $n = 14$ and $k = 5$, exactly

$$\binom{14-5+1}{5} = \binom{10}{5} = 252$$

of them contain no two consecutive numbers. Every five-element subset is of exactly one of the two kinds, so by the Addition Principle $2002 = m + 252$, and $m = 1750$. The remainder when 1750 is divided by 1000 is 750.

Answer: 750. □

Problem 1.3.3 (Adapted from 1983 AIME, Problem #7)**.** Twenty-five of King Arthur's knights are seated at their customary round table. Three of them are chosen and sent off to slay a troublesome dragon. In how many ways can the three knights be chosen so that no two of them had been sitting next to each other?

Strategy. Going around the table, the numbers of unchosen knights between consecutive chosen knights are positive integers with a fixed sum. Count these lists, and then account for the choice of the knight to start from.

Solution. Call one of the three chosen knights K_1, and call the next chosen knights clockwise K_2 and K_3. Let a, b and c be the numbers of unchosen knights between K_1 and K_2, between K_2 and K_3, and between K_3 and K_1, going clockwise. Then

$$a + b + c = 25 - 3 = 22,$$

and no two chosen knights are neighbors exactly when $a, b, c \geq 1$. By Theorem 1.2.2, there are $\binom{21}{2} = 210$ such triples (a, b, c). The seat of K_1 (25 ways) and the triple (a, b, c) determine the seats of all three knights. So there are $25 \cdot 210$ ways to choose three knights, no two neighbors, together with a choice of which of them is K_1. Each choice of three knights is counted 3 times here, once for each choice of K_1, so the number of choices with no two neighbors is

$$\frac{25 \cdot 210}{3} = 1750.$$

Answer: 1750. □

2. Arrangements in a Circle

This section counts arrangements of distinct objects around a circle, where arrangements that differ by a rotation are the same.

2.1. **Circular Permutations.** A **circular permutation** of n distinct objects is an arrangement of them around a circle, where two arrangements are the same when one is a rotation of the other. For example, when A, B and C sit around a table, the clockwise orders ABC, BCA and CAB are the same arrangement.

Theorem 2.1.1. *The number of circular permutations of $n \geq 1$ distinct objects is $(n-1)!$.*

Proof. Choose one object and rotate each arrangement so that this object is at the top. Each circular permutation has exactly one such rotation. The other $n-1$ objects then fill the remaining $n-1$ positions, read clockwise from the top, in $(n-1)!$ ways. □

The same count comes from arrangements in a row. Reading a circular permutation clockwise, starting from each of the n objects in turn, gives n different arrangements in a row, and every arrangement in a row arises from exactly one circular permutation. So the number of circular permutations is $n!/n = (n-1)!$.

Example.

Problem 2.1.2 (2011 AIME I, Problem #5)**.** The vertices of a regular nonagon (9-sided polygon) are to be labeled with the digits 1 through 9 in such a way that the sum of the numbers on every three consecutive vertices is a multiple of 3. Two acceptable arrangements are considered to be indistinguishable if one can be obtained from the other by rotating the nonagon in the plane. Find the number of distinguishable acceptable arrangements.

Strategy. Find the acceptable labelings by looking at remainders when divided by 3. Count them with the vertices held in place, and then account for rotations as in the count of circular permutations.

Solution. Number the vertices 1 to 9 in order around the nonagon, let x_i be the digit at vertex i, and read vertex numbers around the nonagon, so that vertex 10 is vertex 1, vertex 11 is vertex 2, and so on. If $x_i+x_{i+1}+x_{i+2}$ and $x_{i+1}+x_{i+2}+x_{i+3}$ are both multiples of 3, then so is their difference $x_{i+3}-x_i$. So in an acceptable labeling, x_i and x_{i+3} leave the same remainder when divided by 3, for every i. Hence the digits at vertices 1, 4, 7 leave the same remainder, the digits at vertices 2, 5, 8 leave the same remainder, and the digits at vertices 3, 6, 9 leave the same remainder.

Among the digits 1 to 9, the digits 3, 6, 9 leave remainder 0, the digits 1, 4, 7 leave remainder 1, and the digits 2, 5, 8 leave remainder 2. Each of the three groups of vertices holds three digits with the same remainder, so each group of vertices holds one group of digits. Conversely, in such a labeling every three consecutive vertices hold one digit from each group of digits, so their sum leaves the same remainder as $0+1+2=3$ and is a multiple of 3. So a labeling is acceptable exactly when each group of vertices holds one group of digits.

The groups of digits can be assigned to the groups of vertices in $3! = 6$ ways, and within each group the three digits can be placed on the three vertices in $3! = 6$ ways. By the Multiplication Principle there are $6 \cdot 6^3 = 1296$ acceptable labelings with the vertices held in place.

Rotating an acceptable labeling gives an acceptable labeling. A rotation by 1 to 8 steps puts at vertex 1 the digit of another vertex, which is different, so it never gives back the same labeling. Hence the 9 rotations of a labeling are 9 different labelings. As in the count of circular permutations, each distinguishable arrangement corresponds to exactly 9 labelings, and the number of

distinguishable acceptable arrangements is

$$\frac{1296}{9} = 144.$$

Answer: 144. □

2.2. **Necklaces and Bracelets.** Beads on a necklace or a bracelet can also be turned over, which reverses the clockwise order. In such problems an arrangement and its mirror image are the same.

Theorem 2.2.1. *For $n \geq 3$ distinct objects, the number of arrangements around a circle, where two arrangements are the same when one is a rotation or a reflection of the other, is*

$$\frac{(n-1)!}{2}.$$

Proof. Take a circular permutation and an object A in it. Let B be the object just clockwise of A and C the object just counterclockwise of A. Since $n \geq 3$, B and C are different objects. In the mirror image, B is just counterclockwise of A. A rotation does not change which object is just clockwise of A, so the mirror image is a different circular permutation. The mirror image of the mirror image is the original arrangement, so the $(n-1)!$ circular permutations split into pairs, each consisting of an arrangement and its mirror image. Each pair is one arrangement up to rotation and reflection, so there are $(n-1)!/2$ of them. □

The wording of a problem decides which count applies. People around a round table, where clockwise and counterclockwise orders are different, are counted as circular permutations. Beads on a necklace, which can be turned over, are counted up to rotation and reflection.

Example.

Problem 2.2.2 (2014 AMC 10B, Problem #24)**.** The numbers 1, 2, 3, 4, 5 are to be arranged in a circle. An arrangement is *bad* if it is not true that for every n from 1 to 15 one can find a subset of the numbers that appear consecutively on the circle that sum to n. Arrangements that differ only by a rotation or a reflection are considered the same. How many different bad arrangements are there?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

Strategy. By Theorem 2.2.1 there are only 12 arrangements. First find which sums can be missing, and then describe the bad arrangements by which numbers are neighbors.

Solution. By Theorem 2.2.1, there are $(5-1)!/2 = 12$ arrangements in all. Call a set of numbers that appear consecutively on the circle a **block**. Each of 1, 2, 3, 4, 5 is the sum of a block of one number, and 15 is the sum of the block of all five numbers. If a block of one to four numbers has sum s, the other numbers also form a block, and its sum is $15 - s$. So 10, 11, 12, 13 and 14 are block sums in every arrangement, 9 is a block sum exactly when 6 is, and 8 is a block sum exactly when 7 is. Hence an arrangement is bad exactly when 6 or 7 is not a block sum.

A block of four numbers has sum at least $15 - 5 = 10$. A block of two numbers has sum 6 only when it is $\{1, 5\}$ or $\{2, 4\}$, and a block of three numbers has sum 6 only when it is $\{1, 2, 3\}$, which happens exactly when the other two numbers, 4 and 5, are neighbors. So 6 is not a block sum exactly when none of the pairs $\{1, 5\}$, $\{2, 4\}$, $\{4, 5\}$ are neighbors. In that case the neighbors of 5 are 2 and 3, and the neighbors of 4 are 1 and 3, so the numbers around the circle are $2, 5, 3, 4, 1$, in this order. In this arrangement none of the three pairs are neighbors, so it is bad.

In the same way, a block of two numbers has sum 7 only when it is $\{2, 5\}$ or $\{3, 4\}$, and a block of three numbers has sum 7 only when it is $\{1, 2, 4\}$, which happens exactly when 3 and 5 are neighbors. So 7 is not a block sum exactly when none of the pairs $\{2, 5\}$, $\{3, 4\}$, $\{3, 5\}$ are

neighbors. Then the neighbors of 5 are 1 and 4, and the neighbors of 3 are 1 and 2, so the numbers around the circle are $4, 5, 1, 3, 2$, in this order, and this arrangement is bad.

The two bad arrangements are different, because 1 and 2 are neighbors in the first and not in the second. So exactly 2 of the 12 arrangements are bad.

Answer: (B) 2. □

3. Grouping and Partitioning

This section counts the ways to divide a set of distinct objects into groups that do not overlap. The count depends on whether the groups are labeled.

3.1. **Labeled Groups.** Suppose n distinct objects are to be divided into k labeled groups of sizes $n_1, n_2, \ldots, n_k$, where $n_1 + n_2 + \cdots + n_k = n$. Fill the groups one at a time. The first group can be filled in $\binom{n}{n_1}$ ways, the second group, from the remaining $n - n_1$ objects, in $\binom{n-n_1}{n_2}$ ways, and so on; the last group takes the n_k objects that remain, in $\binom{n_k}{n_k} = 1$ way. By the Multiplication Principle, the number of ways is

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n_k}{n_k} = \frac{n!}{n_1!\,(n-n_1)!}\cdot\frac{(n-n_1)!}{n_2!\,(n-n_1-n_2)!}\cdots\frac{n_k!}{n_k!\,0!}.$$

The intermediate factorials cancel.

Theorem 3.1.1. *The number of ways to divide n distinct objects into k labeled groups of sizes $n_1, n_2, \ldots, n_k$, where $n_1 + n_2 + \cdots + n_k = n$, is the* ***multinomial coefficient***

$$\binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1!\,n_2!\cdots n_k!}.$$

This is also the number of arrangements in a row of a multiset with n_1 copies of the first label, n_2 copies of the second label, and so on: writing the label of each object's group under the object turns a division into groups into such an arrangement.

3.2. **Unlabeled Groups.** If the groups are not labeled, the multinomial coefficient overcounts. For example, dividing the 4 people A, B, C, D into two labeled groups of two can be done in $4!/(2!\,2!) = 6$ ways. If the two groups have no names, then the division into $\{\mathrm{A}, \mathrm{B}\}$ and $\{\mathrm{C}, \mathrm{D}\}$ is the same as the division into $\{\mathrm{C}, \mathrm{D}\}$ and $\{\mathrm{A}, \mathrm{B}\}$, so each division is counted twice, and there are only 3 divisions.

Theorem 3.2.1. *When the groups are nonempty and not labeled, the number of divisions is the multinomial coefficient divided by $m!$ for each collection of m groups that have the same size.*

Proof. A division into unlabeled groups becomes a division into labeled groups when the labels are given to the groups. Each label is attached to a group size, so only groups of equal size can exchange labels. If m groups have the same size, they are m different sets, and their m labels can be given to them in $m!$ ways. By the Multiplication Principle, each division into unlabeled groups gives exactly as many divisions into labeled groups as the product of these factorials, and every division into labeled groups arises from exactly one division into unlabeled groups. □

Example.

Problem 3.2.2. In how many ways can a group of 9 distinct students be split into 3 study groups of 3 students each?

Strategy. First split the students into three labeled groups with the multinomial coefficient. The study groups have no names and equal sizes, so then divide by the number of ways to order the three groups.

Solution. By Theorem 3.1.1, the number of ways to split the students into three labeled groups of 3 is

$$\binom{9}{3,3,3} = \frac{9!}{3!\,3!\,3!} = \frac{362{,}880}{216} = 1680.$$

The study groups are not labeled and all three have the same size, so by Theorem 3.2.1 each split is counted $3! = 6$ times. The number of splits is

$$\frac{1680}{6} = 280.$$

Answer: 280. □

4. The Principle of Inclusion-Exclusion

For a finite set A, $|A|$ denotes its number of elements. The **union** $A \cup B$ is the set of elements in A or B (or both), and the **intersection** $A \cap B$ is the set of elements in both A and B. Inside a fixed set S, the **complement** A^c is the set of elements of S that are not in A, so $|A^c| = |S| - |A|$.

Theorem 4.1 (Inclusion-Exclusion)**.** *For finite sets A, B and C,*

$$|A \cup B| = |A| + |B| - |A \cap B|,$$

$$|A \cup B \cup C| = |A| + |B| + |C| - \big(|A \cap B| + |A \cap C| + |B \cap C|\big) + |A \cap B \cap C|.$$

In general, for finite sets $A_1, A_2, \ldots, A_n$,

$$\Big|\bigcup_{i=1}^{n} A_i\Big| = \sum_i |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<l} |A_i \cap A_j \cap A_l| - \cdots + (-1)^{n-1} |A_1 \cap A_2 \cap \cdots \cap A_n|.$$

Proof. An element in none of the sets is counted 0 times on the right side. Suppose an element lies in exactly $t \geq 1$ of the sets. It is counted $\binom{t}{1}$ times in the first sum, $\binom{t}{2}$ times in the second sum, and so on, and 0 times in each sum over more than t sets, so the right side counts it

$$\binom{t}{1} - \binom{t}{2} + \binom{t}{3} - \cdots + (-1)^{t-1}\binom{t}{t}$$

times. By the binomial theorem, $0 = (1-1)^t = 1 - \binom{t}{1} + \binom{t}{2} - \cdots + (-1)^t\binom{t}{t}$, so this number is 1. Hence the right side counts each element of the union exactly once. The formulas for two and three sets are the cases $n = 2$ and $n = 3$. □

Example.

Problem 4.2 (2001 AMC 12, Problem #12)**.** How many positive integers not exceeding 2001 are multiples of 3 or 4 but not 5?

(A) 768 **(B)** 801 **(C)** 934 **(D)** 1067 **(E)** 1167

Strategy. Count the multiples of 3 or 4 with inclusion-exclusion, then subtract those that are also multiples of 5. A number is a multiple of both 3 and 4 exactly when it is a multiple of 12.

Solution. Let $S = \{1, 2, \ldots, 2001\}$, and for a positive integer m let A_m be the set of multiples of m in S, so that $|A_m| = \lfloor 2001/m \rfloor$. No two of 3, 4 and 5 have a common factor greater than 1, so a number is a multiple of two of them exactly when it is a multiple of their product. So $A_3 \cap A_4 = A_{12}$, $A_3 \cap A_5 = A_{15}$ and $A_4 \cap A_5 = A_{20}$. In the same way, a number is a multiple of both 15 and 20 exactly when it is a multiple of 3, 4 and 5, that is, of 60, so $A_{15} \cap A_{20} = A_{60}$.

By Theorem 4.1, the number of multiples of 3 or 4 is

$$|A_3 \cup A_4| = |A_3| + |A_4| - |A_{12}| = 667 + 500 - 166 = 1001.$$

Among these, the multiples of 5 are the numbers in $A_{15} \cup A_{20}$, and there are

$$|A_{15} \cup A_{20}| = |A_{15}| + |A_{20}| - |A_{60}| = 133 + 100 - 33 = 200$$

of them. So the number of multiples of 3 or 4 that are not multiples of 5 is $1001 - 200 = 801$.

Answer: (B) 801. □

5. The Hockey-Stick Identity

The hockey-stick identity gives the sum of the entries along a diagonal of Pascal's triangle. Such sums arise when a stars and bars count is split into cases by the value of one quantity.

Theorem 5.1 (Hockey-Stick Identity)**.** *For integers $n \geq r \geq 0$,*

$$\sum_{k=r}^{n} \binom{k}{r} = \binom{r}{r} + \binom{r+1}{r} + \cdots + \binom{n}{r} = \binom{n+1}{r+1}.$$

Proof. For $m = r, r+1, \ldots, n$, let $S_m = \binom{r}{r} + \binom{r+1}{r} + \cdots + \binom{m}{r}$ be the sum of the terms up to $\binom{m}{r}$. Then $S_r = \binom{r}{r} = 1 = \binom{r+1}{r+1}$. If $S_m = \binom{m+1}{r+1}$ for some $m < n$, then Pascal's identity gives

$$S_{m+1} = S_m + \binom{m+1}{r} = \binom{m+1}{r+1} + \binom{m+1}{r} = \binom{m+2}{r+1}.$$

Applying this for $m = r, r+1, \ldots, n-1$ in turn shows that $S_m = \binom{m+1}{r+1}$ for every m from r to n. In particular, $S_n = \binom{n+1}{r+1}$. □

The name comes from the shape of these entries in the triangle: the terms of the sum form the stick, and the result is the blade. For $r = 1$ the identity is the formula

$$1 + 2 + \cdots + n = \binom{n+1}{2} = \frac{n(n+1)}{2}.$$

Example.

Problem 5.2. How many ordered triples (a, b, c) of nonnegative integers satisfy $a + b + c \leq 10$?

Strategy. Split the triples into cases by the value of $a + b + c$. Each case is a stars and bars count, and these counts lie along a diagonal of Pascal's triangle.

Solution. For each $s = 0, 1, \ldots, 10$, the triples with $a + b + c = s$ are the nonnegative integer solutions of this equation, so by Theorem 1.2.1 with 3 variables there are $\binom{s+2}{2}$ of them. Each triple with $a + b + c \leq 10$ is counted for exactly one value of s, so by the Addition Principle the number of triples is

$$\sum_{s=0}^{10} \binom{s+2}{2} = \binom{2}{2} + \binom{3}{2} + \cdots + \binom{12}{2}.$$

By Theorem 5.1 with $r = 2$ and $n = 12$, this sum is

$$\binom{13}{3} = \frac{13 \cdot 12 \cdot 11}{3 \cdot 2 \cdot 1} = 286.$$

As a check, set $d = 10 - a - b - c$. The triples correspond one to one to the nonnegative integer solutions of $a + b + c + d = 10$, and Theorem 1.2.1 with 4 variables gives $\binom{13}{3} = 286$ again.

Answer: 286. □

6. Problems

Beginner.

1. (2001 AMC 10, Problem #19) Pat wants to buy four donuts from an ample supply of three types of donuts: glazed, chocolate, and powdered. How many different selections are possible?

(A) 6 **(B)** 9 **(C)** 12 **(D)** 15 **(E)** 18

2. (1958 AHSME, Problem #49) In the expansion of $(a+b)^n$ there are $n+1$ dissimilar terms. The number of dissimilar terms in the expansion of $(a+b+c)^{10}$ is:

(A) 11 **(B)** 33 **(C)** 55 **(D)** 66 **(E)** 132

3. (Adapted from 2019 AMC 8, Problem #25) In how many ways can 10 identical apples be distributed among Alice, Becky, and Chris so that each person has at least one apple?

4. Five boys and five girls are to sit at a round table with ten seats so that boys and girls alternate. Two seatings are considered the same if one is a rotation of the other. How many different seatings are possible?

5. How many ways can 8 people be seated around a circular table if two arrangements are considered the same whenever each person has the same two neighbors?

Intermediate.

6. (1989 AHSME, Problem #27) Let n be a positive integer. If the equation $2x+2y+z=n$ has 28 solutions in positive integers x, y and z, then n must be either

(A) 14 or 15 **(B)** 15 or 16 **(C)** 16 or 17 **(D)** 17 or 18 **(E)** 18 or 19

7. How many integers from 1 to 1000, inclusive, are divisible by 2, 3, or 5?

8. (2000 AIME II, Problem #5) Given eight distinguishable rings, let n be the number of possible five-ring arrangements on the four fingers (not the thumb) of one hand. The order of rings on each finger is significant, but it is not required that each finger have a ring. Find the leftmost three nonzero digits of n.

9. (2014 AMC 12A, Problem #13) A fancy bed and breakfast inn has 5 rooms, each with a distinctive color-coded decor. One day 5 friends arrive to spend the night. There are no other guests that night. The friends can room in any combination they wish, but with no more than 2 friends per room. In how many ways can the innkeeper assign the guests to the rooms?

(A) 2100 **(B)** 2220 **(C)** 3000 **(D)** 3120 **(E)** 3125

10. (Adapted from 2014 AIME II, Problem #9) Ten chairs are arranged in a circle. In how many ways can four of the chairs be chosen so that no two chosen chairs are adjacent?

Advanced.

11. (2008 AMC 10A, Problem #23) Two subsets of the set $S=\{a,b,c,d,e\}$ are to be chosen so that their union is S and their intersection contains exactly two elements. In how many ways can this be done, assuming that the order in which the subsets are chosen does not matter?

(A) 20 **(B)** 40 **(C)** 60 **(D)** 160 **(E)** 320

12. (2021 AIME I, Problem #4) Find the number of ways 66 identical coins can be separated into three nonempty piles so that there are fewer coins in the first pile than in the second pile and fewer coins in the second pile than in the third pile.

13. How many ways are there to arrange the letters A, A, B, B, C, C, D, D in a row so that no two adjacent letters are identical?

14. (2020 AMC 10B, Problem #25) Let $D(n)$ denote the number of ways of writing the positive integer n as a product

$$n = f_1 \cdot f_2 \cdots f_k,$$

where $k \geq 1$, the f_i are integers strictly greater than 1, and the order in which the factors are listed matters (that is, two representations that differ only in the order of the factors are counted as distinct). For example, the number 6 can be written as 6, $2 \cdot 3$, and $3 \cdot 2$, so $D(6) = 3$. What is $D(96)$?

(A) 112 **(B)** 128 **(C)** 144 **(D)** 172 **(E)** 184

15. (2006 AMC 12A, Problem #25) How many non-empty subsets S of $\{1, 2, 3, \ldots, 15\}$ have the following two properties?
(1) No two consecutive integers belong to S.
(2) If S contains k elements, then S contains no number less than k.

(A) 277 **(B)** 311 **(C)** 376 **(D)** 377 **(E)** 405