STARS AND BARS, DISTRIBUTIONS, AND CIRCULAR ARRANGEMENTS FOR AMC 10: SOLUTIONS

ANDY JO

Contents

1. Solutions 2
2. Answer Key 9

1. Solutions

Beginner.

1. (2001 AMC 10, Problem #19) Pat wants to buy four donuts from an ample supply of three types of donuts: glazed, chocolate, and powdered. How many different selections are possible?

(A) 6 **(B)** 9 **(C)** 12 **(D)** 15 **(E)** 18

Strategy. A selection is a combination with repetition: 4 donuts chosen from 3 types.

Solution. A selection is determined by how many donuts of each type it contains, and the order in which the donuts are bought does not matter. So a selection is a choice of 4 items from 3 types of items, where a type may be chosen more than once. By the count of combinations with repetition (stars and bars) with $n = 3$ and $k = 4$, the number of selections is

$$\binom{3+4-1}{4} = \binom{6}{4} = 15.$$

Answer: (D) 15. □

2. (1958 AHSME, Problem #49) In the expansion of $(a+b)^n$ there are $n+1$ dissimilar terms. The number of dissimilar terms in the expansion of $(a+b+c)^{10}$ is:

(A) 11 **(B)** 33 **(C)** 55 **(D)** 66 **(E)** 132

Strategy. A term of the expansion is determined by its exponents of a, b and c, which are nonnegative integers with sum 10.

Solution. Multiplying out $(a+b+c)^{10}$ gives a sum of products of 10 factors, each of which is a, b or c. A product with i factors a, j factors b and k factors c equals $a^i b^j c^k$, where i, j and k are nonnegative integers with $i+j+k = 10$, and every such list of exponents occurs. All the coefficients are positive, so after like terms are combined there is exactly one term for each list (i, j, k). By the count of nonnegative integer solutions (stars and bars) with 3 variables and sum 10, the number of dissimilar terms is

$$\binom{10+3-1}{3-1} = \binom{12}{2} = 66.$$

Answer: (D) 66. □

3. (Adapted from 2019 AMC 8, Problem #25) In how many ways can 10 identical apples be distributed among Alice, Becky, and Chris so that each person has at least one apple?

Strategy. The apples are identical, so a distribution is determined by the number of apples each person receives. These three numbers are positive integers with sum 10.

Solution. Let a, b and c be the numbers of apples that Alice, Becky and Chris receive. Each distribution gives exactly one solution of

$$a + b + c = 10, \qquad a, b, c \geq 1,$$

and each solution gives exactly one distribution. By the count of positive integer solutions (stars and bars) with 3 variables and sum 10, the number of solutions is

$$\binom{10-1}{3-1} = \binom{9}{2} = 36.$$

Answer: 36. □

4. Five boys and five girls are to sit at a round table with ten seats so that boys and girls alternate. Two seatings are considered the same if one is a rotation of the other. How many different seatings are possible?

Strategy. Seat the boys first. Their order around the table is a circular permutation, and each of the five seats between two neighboring boys holds one girl.

Solution. In a seating, read the boys clockwise around the table. Their order is a circular permutation of the five boys, so by the count of circular permutations it can be chosen in $(5-1)! = 24$ ways. Because boys and girls alternate, exactly one girl sits just clockwise of each boy. A rotation does not change the clockwise order of the boys or which girl sits just clockwise of each boy. Conversely, these determine the seating up to rotation: going clockwise from any boy, the people are that boy, the girl just clockwise of him, the next boy, the girl just clockwise of that boy, and so on. Once the order of the boys is fixed, the five girls can be placed just clockwise of the five boys in $5! = 120$ ways. By the Multiplication Principle, the number of seatings is $24 \cdot 120 = 2880$.

As a check, if the ten seats are numbered, the boys take either the odd-numbered seats or the even-numbered seats (2 ways), and the boys and the girls fill their seats in $5! \cdot 5!$ ways. This gives $2 \cdot 120 \cdot 120 = 28{,}800$ seatings at a table with numbered seats. Each seating of the round table corresponds to exactly 10 of these, one for each rotation, and $28{,}800/10 = 2880$.

Answer: 2880. □

5. How many ways can 8 people be seated around a circular table if two arrangements are considered the same whenever each person has the same two neighbors?

Strategy. Show that two seatings give every person the same two neighbors exactly when one is a rotation or a reflection of the other. Then count arrangements around a circle up to rotation and reflection.

Solution. Rotating or reflecting a seating does not change anyone's neighbors. Conversely, suppose two seatings give every person the same two neighbors. Fix a person A, with neighbors B and C. Going around the table from A toward B, each next person is the neighbor of the current person other than the one just passed. So the neighbors determine the whole order around the table once the first step from A is fixed. If the clockwise neighbor of A is the same person in both seatings, the two seatings have the same clockwise order and are rotations of each other. Otherwise the clockwise order of one is the counterclockwise order of the other, and the seatings are mirror images.

So the answer is the number of arrangements of 8 distinct people around a circle, where arrangements that differ by a rotation or a reflection are the same. By the count for necklaces with $n = 8$, this is

$$\frac{(8-1)!}{2} = \frac{5040}{2} = 2520.$$

Answer: 2520. □

Intermediate.

6. (1989 AHSME, Problem #27) Let n be a positive integer. If the equation $2x + 2y + z = n$ has 28 solutions in positive integers x, y and z, then n must be either

(A) 14 or 15 **(B)** 15 or 16 **(C)** 16 or 17 **(D)** 17 or 18 **(E)** 18 or 19

Strategy. Group the solutions by the value of $x + y$. For each value, count the pairs (x, y) with stars and bars, and add the counts with the hockey-stick identity.

Solution. Let (x, y, z) be a solution in positive integers, and let $s = x + y$. Then $s \geq 2$ and $z = n - 2s$, which is positive exactly when $2s \leq n - 1$. Let m be the largest integer with $2m \leq n - 1$. The possible values of s are $2, 3, \ldots, m$, and for each of them every pair of positive integers (x, y) with $x + y = s$ gives exactly one solution, with $z = n - 2s$. By the count of positive integer solutions with 2 variables, there are $\binom{s-1}{1} = s - 1$ such pairs. So if $m \geq 2$, the number of solutions is

$$\binom{1}{1} + \binom{2}{1} + \cdots + \binom{m-1}{1} = \binom{m}{2}$$

by the hockey-stick identity with $r = 1$, and if $m < 2$ there are no solutions. The equation has 28 solutions exactly when $\binom{m}{2} = \frac{m(m-1)}{2} = 28$, that is, $m(m-1) = 56$. The product $m(m-1)$ increases with m for $m \geq 1$, and $8 \cdot 7 = 56$, so $m = 8$. Finally, $m = 8$ means $2 \cdot 8 \leq n - 1 < 2 \cdot 9$, that is, $17 \leq n < 19$. So n is 17 or 18.

Answer: (D) 17 or 18. □

7. How many integers from 1 to 1000, inclusive, are divisible by 2, 3, or 5?

Strategy. Use inclusion-exclusion for three sets. Since 2, 3 and 5 are primes, an integer is divisible by two or three of them exactly when it is divisible by their product.

Solution. Let A, B and C be the sets of integers from 1 to 1000 that are divisible by 2, 3 and 5. The number of multiples of m from 1 to 1000 is $\lfloor 1000/m \rfloor$, so

$$|A| = 500, \quad |B| = 333, \quad |C| = 200,$$

$$|A \cap B| = \left\lfloor \frac{1000}{6} \right\rfloor = 166, \quad |A \cap C| = \left\lfloor \frac{1000}{10} \right\rfloor = 100, \quad |B \cap C| = \left\lfloor \frac{1000}{15} \right\rfloor = 66,$$

and $|A \cap B \cap C| = \lfloor 1000/30 \rfloor = 33$. By inclusion-exclusion,

$$\begin{aligned} |A \cup B \cup C| &= (500 + 333 + 200) - (166 + 100 + 66) + 33 \\ &= 1033 - 332 + 33 = 734. \end{aligned}$$

Answer: 734. □

8. (2000 AIME II, Problem #5) Given eight distinguishable rings, let n be the number of possible five-ring arrangements on the four fingers (not the thumb) of one hand. The order of rings on each finger is significant, but it is not required that each finger have a ring. Find the leftmost three nonzero digits of n.

Strategy. Choose the five rings. Then record an arrangement as a row of the five rings and three bars that separate the four fingers.

Solution. The five rings that are used can be chosen in $\binom{8}{5} = 56$ ways. Fix the five rings. List the rings on the first finger from the base of the finger outward, then write a bar, then list the rings on the second finger in the same way, then a bar, then the rings on the third finger, a bar, and the rings on the fourth finger. This gives a row of 8 symbols, the 5 rings and 3 bars. Each arrangement gives exactly one such row, and each row gives back exactly one arrangement: the bars split the rings into the four fingers in order, and the order of the rings on each finger is their order in the row. A row is made by choosing the 3 positions of the bars, in $\binom{8}{3} = 56$ ways, and then placing the 5 rings in the other 5 positions, in $5! = 120$ ways. By the Multiplication Principle,

$$n = 56 \cdot 56 \cdot 120 = 376{,}320,$$

and the leftmost three nonzero digits of n are 376.

Answer: 376. □

9. (2014 AMC 12A, Problem #13) A fancy bed and breakfast inn has 5 rooms, each with a distinctive color-coded decor. One day 5 friends arrive to spend the night. There are no other guests that night. The friends can room in any combination they wish, but with no more than 2 friends per room. In how many ways can the innkeeper assign the guests to the rooms?

(A) 2100 **(B)** 2220 **(C)** 3000 **(D)** 3120 **(E)** 3125

Strategy. Split into cases by how many rooms hold two friends. In each case, divide the friends into groups and send the groups to different rooms.

Solution. Each room holds 0, 1 or 2 friends, and the five friends fill the rooms. So the occupied rooms hold $1, 1, 1, 1, 1$ friends, or $2, 1, 1, 1$, or $2, 2, 1$.

If every friend is alone, the five friends go to the five rooms in $5! = 120$ ways.

If exactly one room holds two friends, the pair can be chosen in $\binom{5}{2} = 10$ ways. The pair and the three other friends form four groups, which go to four different rooms in $5 \cdot 4 \cdot 3 \cdot 2 = 120$ ways. This case gives $10 \cdot 120 = 1200$ ways.

If two rooms hold two friends each, the friend who is alone can be chosen in 5 ways. The other four friends form two pairs. The pairs are not labeled, so by the rule for unlabeled groups there are $\binom{4}{2,2}/2! = 6/2 = 3$ ways to form them. The three groups go to three different rooms in $5 \cdot 4 \cdot 3 = 60$ ways. This case gives $5 \cdot 3 \cdot 60 = 900$ ways.

The total is $120 + 1200 + 900 = 2220$.

Answer: (B) 2220. □

10. (Adapted from 2014 AIME II, Problem #9) Ten chairs are arranged in a circle. In how many ways can four of the chairs be chosen so that no two chosen chairs are adjacent?

Strategy. Split into cases by whether one particular chair is chosen. In each case the chairs that can still be chosen form a row.

Solution. Number the chairs 1 to 10 around the circle, so that chair 10 is next to chair 1.

If chair 1 is chosen, chairs 2 and 10 cannot be chosen. The other three chosen chairs come from chairs 3 to 9, which form a row of 7 chairs, and no two of them may be adjacent. By the count of non-adjacent selections, there are $\binom{7-3+1}{3} = \binom{5}{3} = 10$ ways.

If chair 1 is not chosen, the four chosen chairs come from chairs 2 to 10. These form a row of 9 chairs, since chairs 2 and 10 are not next to each other. There are $\binom{9-4+1}{4} = \binom{6}{4} = 15$ ways.

The total is $10 + 15 = 25$.

Answer: 25. □

Advanced.

11. (2008 AMC 10A, Problem #23) Two subsets of the set $S = \{a, b, c, d, e\}$ are to be chosen so that their union is S and their intersection contains exactly two elements. In how many ways can this be done, assuming that the order in which the subsets are chosen does not matter?

(A) 20 **(B)** 40 **(C)** 60 **(D)** 160 **(E)** 320

Strategy. First count ordered pairs of subsets, as if the two subsets were labeled. Then correct for the fact that their order does not matter, as with unlabeled groups.

Solution. Call the two subsets A and B, for now in this order. Since $A \cup B = S$, each element of S lies in A only, in B only, or in both. Exactly two elements lie in both subsets, and they can be chosen in $\binom{5}{2} = 10$ ways. Each of the other three elements lies in exactly

one of A and B, so they can be placed in $2^3 = 8$ ways. This gives $10 \cdot 8 = 80$ ordered pairs (A, B).

The two subsets are never equal, because each of the three other elements lies in only one of them. So each unordered choice of two subsets is counted exactly twice among the ordered pairs, once as (A, B) and once as (B, A). The number of ways is

$$\frac{80}{2} = 40.$$

Answer: (B) 40. □

12. (2021 AIME I, Problem #4) Find the number of ways 66 identical coins can be separated into three nonempty piles so that there are fewer coins in the first pile than in the second pile and fewer coins in the second pile than in the third pile.

Strategy. Count the ordered triples of positive pile sizes with sum 66, remove those with two equal sizes by inclusion-exclusion, and then divide by the number of orders of three different numbers.

Solution. Let the three piles have a, b and c coins, in order. The required separations are the positive integer solutions of $a + b + c = 66$ with $a < b < c$.

By the count of positive integer solutions (stars and bars), the equation $a + b + c = 66$ has $\binom{65}{2} = 2080$ positive integer solutions. Let E_1, E_2 and E_3 be the sets of these solutions with $a = b$, with $b = c$, and with $a = c$. A solution with $a = b$ has $c = 66 - 2a \geq 1$, so a is one of $1, 2, \ldots, 32$, and $|E_1| = 32$. In the same way $|E_2| = |E_3| = 32$. A solution in two of the sets has $a = b = c = 22$, so each intersection of two or three of the sets contains only this solution. By inclusion-exclusion,

$$|E_1 \cup E_2 \cup E_3| = 3 \cdot 32 - 3 \cdot 1 + 1 = 94.$$

So $2080 - 94 = 1986$ solutions have three different numbers.

Each set of three different positive integers with sum 66 gives $3! = 6$ of these solutions, one for each order, and exactly one of the six orders is increasing. So the number of solutions with $a < b < c$ is

$$\frac{1986}{6} = 331.$$

Answer: 331. □

13. How many ways are there to arrange the letters A, A, B, B, C, C, D, D in a row so that no two adjacent letters are identical?

Strategy. Count all arrangements, then use inclusion-exclusion to subtract the arrangements in which at least one pair of identical letters is adjacent.

Solution. Without the condition, the letters have

$$\frac{8!}{2!\,2!\,2!\,2!} = \frac{40{,}320}{16} = 2520$$

arrangements. Let P_A be the set of arrangements in which the two A's are adjacent, and define P_B, P_C and P_D in the same way. The answer is $2520 - |P_A \cup P_B \cup P_C \cup P_D|$.

In an arrangement in P_A, glue the two A's into one block AA. The arrangements in P_A correspond one to one to the arrangements of the seven objects AA, B, B, C, C, D, D, so

$$|P_A| = \frac{7!}{2!\,2!\,2!} = 630.$$

In the same way, gluing two pairs gives $|P_A \cap P_B| = 6!/(2!\,2!) = 180$, gluing three pairs gives $|P_A \cap P_B \cap P_C| = 5!/2! = 60$, and gluing all four pairs gives $|P_A \cap P_B \cap P_C \cap P_D| = 4! = 24$. Renaming the letters shows that every intersection of j of the four sets has the same size as the one computed, and there are $\binom{4}{j}$ such intersections. By inclusion-exclusion,

$$\begin{aligned}|P_A \cup P_B \cup P_C \cup P_D| &= 4 \cdot 630 - 6 \cdot 180 + 4 \cdot 60 - 24 \\ &= 2520 - 1080 + 240 - 24 = 1656.\end{aligned}$$

So the number of arrangements with no two adjacent letters identical is $2520 - 1656 = 864$.

Answer: 864. □

14. (2020 AMC 10B, Problem #25) Let $D(n)$ denote the number of ways of writing the positive integer n as a product

$$n = f_1 \cdot f_2 \cdots f_k,$$

where $k \geq 1$, the f_i are integers strictly greater than 1, and the order in which the factors are listed matters (that is, two representations that differ only in the order of the factors are counted as distinct). For example, the number 6 can be written as 6, $2 \cdot 3$, and $3 \cdot 2$, so $D(6) = 3$. What is $D(96)$?

(A) 112 **(B)** 128 **(C)** 144 **(D)** 172 **(E)** 184

Strategy. Fix the number k of factors. Count the possible positions of the factor divisible by 3, and count the possible lists of exponents of 2 in the k factors with stars and bars.

Solution. Since $96 = 2^5 \cdot 3$, every factor is a power of 2 or is 3 times a power of 2, and exactly one factor is divisible by 3. In an ordered factorization with k factors, the position of that factor can be chosen in k ways. Write that factor as $3 \cdot 2^a$ with $a \geq 0$, and write each other factor as 2^{a_i} with $a_i \geq 1$, since those factors are greater than 1.

The exponents of 2 add up to 5. Subtracting 1 from each of the $k-1$ exponents a_i leaves k nonnegative integers, a and the numbers $a_i - 1$, with sum $5 - (k-1) = 6 - k$. This requires $1 \leq k \leq 6$, and by the count of nonnegative integer solutions (stars and bars) there are

$$\binom{(6-k)+k-1}{k-1} = \binom{5}{k-1}$$

choices of the exponents. Each factorization determines its number of factors, the position of the factor divisible by 3 and all the exponents, so

$$D(96) = \sum_{k=1}^{6} k\binom{5}{k-1} = 1 + 10 + 30 + 40 + 25 + 6 = 112.$$

Answer: (A) 112. □

15. (2006 AMC 12A, Problem #25) How many non-empty subsets S of $\{1, 2, 3, \ldots, 15\}$ have the following two properties?

(1) No two consecutive integers belong to S.

(2) If S contains k elements, then S contains no number less than k.

(A) 277 **(B)** 311 **(C)** 376 **(D)** 377 **(E)** 405

Strategy. Split into cases by the number of elements k. Condition (2) limits the elements to the integers from k to 15, and condition (1) then makes each case a selection of non-adjacent objects in a row.

Solution. Suppose S has k elements. By condition (2), all of them come from the $16-k$ integers $k, k+1, \ldots, 15$. By condition (1), no two of them are consecutive, so S is a choice of k objects, no two adjacent, from $16-k$ objects in a row. If $k \geq 9$, then $16-k < k$, and there is no such choice. If $k \leq 8$, then $k \leq 16-k$, and by the count of non-adjacent selections there are

$$\binom{(16-k)-k+1}{k} = \binom{17-2k}{k}$$

such subsets. This is 0 when $17-2k < k$, that is, when $k \geq 6$. Adding the cases $k = 1, 2, 3, 4, 5$ gives

$$\binom{15}{1} + \binom{13}{2} + \binom{11}{3} + \binom{9}{4} + \binom{7}{5} = 15 + 78 + 165 + 126 + 21 = 405.$$

Answer: (E) 405. □

2. Answer Key

Problem	Letter	Value
1	D	15
2	D	66
3		36
4		2880
5		2520
6	D	17 or 18
7		734
8		376
9	B	2220
10		25
11	B	40
12		331
13		864
14	A	112
15	E	405