STARS AND BARS, DISTRIBUTIONS, AND CIRCULAR ARRANGEMENTS FOR AMC 10: PRACTICE SET SOLUTIONS

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1. Solutions

Beginner.

1. (2004 AMC 10B, Problem #2) How many two-digit positive integers have at least one 7 as a digit?

(A) 10 **(B)** 18 **(C)** 19 **(D)** 20 **(E)** 30

Strategy. Count the numbers whose tens digit is 7 and the numbers whose units digit is 7. The number 77 is in both groups, so combine the two counts with inclusion-exclusion.

Solution. Let A be the set of two-digit positive integers whose tens digit is 7, and let B be the set of those whose units digit is 7. A two-digit positive integer has at least one 7 as a digit exactly when it lies in $A \cup B$. The set A consists of $70, 71, \ldots, 79$, so $|A| = 10$. The tens digit of a two-digit integer is one of $1, 2, \ldots, 9$, so B consists of $17, 27, \ldots, 97$, and $|B| = 9$. The only integer in both sets is 77, so $|A \cap B| = 1$. By inclusion-exclusion,

$$|A \cup B| = |A| + |B| - |A \cap B| = 10 + 9 - 1 = 18.$$

Answer: (B) 18. □

2. Identical balls are stacked in the shape of a triangular pyramid. The top layer is a single ball, and each layer below it is a triangle with one more ball on each side than the layer above it, so the layers contain 1, 3, 6, 10, ... balls. The bottom layer has 20 balls along each side. How many balls are in the pyramid?

Strategy. Count the balls in each layer first: a triangle with k balls on each side has rows of $1, 2, \ldots, k$ balls. The layer sizes are then entries along a diagonal of Pascal's triangle, and the hockey-stick identity adds them.

Solution. The top layer has 1 ball on each side, and each layer has one more ball on each side than the layer above it. So the kth layer from the top has k balls along each side, and the pyramid has 20 layers. The kth layer is a triangle whose rows have $1, 2, \ldots, k$ balls, so by the hockey-stick identity with $r = 1$ it holds

$$1 + 2 + \cdots + k = \binom{k+1}{2}$$

balls. For $k = 1, 2, 3, 4$ this gives 1, 3, 6 and 10, as in the statement. Adding over the 20 layers, the number of balls in the pyramid is

$$\binom{2}{2} + \binom{3}{2} + \cdots + \binom{21}{2}.$$

By the hockey-stick identity with $r = 2$ and $n = 21$, this sum is

$$\binom{22}{3} = \frac{22 \cdot 21 \cdot 20}{3 \cdot 2 \cdot 1} = 1540.$$

Answer: 1540. □

3. Five married couples are to be seated at a round table with ten chairs so that each person sits next to his or her spouse. Two seatings are considered the same if one is a rotation of the other. How many different seatings are possible?

Strategy. Glue each couple into one block. The five blocks sit around the table in a circular permutation, and the two spouses in each block can sit in either order.

Solution. Each person sits next to his or her spouse, so each couple takes two neighboring chairs. Glue the two spouses of each couple into one block. Reading a seating clockwise around the table gives the order of the five blocks around the table and, in each block, the spouse who comes first. A rotation changes neither of these. Conversely, they give the clockwise order of all ten people around the table, which determines the seating up to rotation. So the seatings correspond one to one to the choices of an order of the blocks around the table and an order inside each block.

The five blocks are different, so by the count of circular permutations their order around the table can be chosen in $(5-1)! = 24$ ways. The spouses in each of the five blocks can sit in 2 orders, which gives $2^5 = 32$ ways. By the Multiplication Principle, the number of seatings is

$$24 \cdot 32 = 768.$$

As a check, number the chairs 1 to 10 around the table. The couples then take either chairs 1 and 2, 3 and 4, and so on, or chairs 2 and 3, 4 and 5, and so on up to chairs 10 and 1, so the chairs can be paired up in 2 ways. The five couples can be given the five pairs of chairs in $5! = 120$ ways, and each couple can sit in its pair in 2 orders, so there are $2 \cdot 120 \cdot 32 = 7680$ seatings in numbered chairs. The 10 rotations of such a seating put 10 different people in chair 1, so they are 10 different seatings, and $7680/10 = 768$.

Answer: 768. □

4. (2017 AMC 10B, Problem #13) There are 20 students participating in an after-school program offering classes in yoga, bridge, and painting. Each student must take at least one of these three classes, but may take two or all three.
There are 10 students taking yoga, 13 taking bridge, and 9 taking painting. There are 9 students taking at least two classes. How many students are taking all three classes?
(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

Strategy. Add the three class sizes. In this sum, a student who takes one class is counted once, a student who takes two classes is counted twice, and a student who takes all three is counted three times.

Solution. Let a, b, and c be the numbers of students who take exactly one, exactly two, and all three classes. Every student takes at least one class, so $a + b + c = 20$. Since 9 students take at least two classes, $b + c = 9$, and therefore $a = 11$.

Adding the class sizes counts each student once for every class the student takes:

$$a + 2b + 3c = 10 + 13 + 9 = 32.$$

Substituting $a = 11$ gives $2b + 3c = 21$. Since $b = 9 - c$, this becomes $18 - 2c + 3c = 21$, so $c = 3$.

Answer: (C) 3. □

5. Ten lockers stand in a row. Some of them, possibly none, are to be painted red so that no two neighboring lockers are both red. In how many ways can the set of red lockers be chosen?

Strategy. Split the choices by the number k of red lockers. For each k, the red lockers are k of 10 objects in a row, no two adjacent, so the count of non-adjacent selections gives the number of choices.

Solution. Suppose exactly k lockers are red. They are k of the 10 lockers in the row, and no two of them are adjacent. By the count of non-adjacent selections with $n = 10$, there

are

$$\binom{10-k+1}{k} = \binom{11-k}{k}$$

such sets of red lockers. This number is 0 when $k > 11-k$, that is, when $k \geq 6$. For $k = 0, 1, 2, 3, 4, 5$ it is

$$\binom{11}{0} = 1, \quad \binom{10}{1} = 10, \quad \binom{9}{2} = 36, \quad \binom{8}{3} = 56, \quad \binom{7}{4} = 35, \quad \binom{6}{5} = 6.$$

Each set of red lockers has exactly one size, so by the Addition Principle the number of sets is

$$1 + 10 + 36 + 56 + 35 + 6 = 144.$$

Answer: 144. □

Intermediate.

6. (2023 AMC 10B, Problem #11) Suzanne went to the bank and withdrew \$800. The teller gave her this amount using \$20 bills, \$50 bills, and \$100 bills, with at least one of each denomination. How many different collections of bills could Suzanne have received?

(A) 45 **(B)** 21 **(C)** 36 **(D)** 28 **(E)** 32

Strategy. Write an equation for the numbers of bills. After simplifying, the count becomes a count of positive integer solutions of a sum, which stars and bars handles.

Solution. Let x, y, and z be the numbers of \$20, \$50, and \$100 bills. Each is at least 1, and

$$20x + 50y + 100z = 800, \quad \text{that is,} \quad 2x + 5y + 10z = 80.$$

The numbers $2x$, $10z$, and 80 are even, so $5y$ is even and y is even. Write $y = 2w$ with $w \geq 1$. Dividing the equation by 2 gives

$$x + 5w + 5z = 40, \quad \text{so} \quad x = 40 - 5(w + z).$$

Let $s = w+z$. Then $x = 40-5s$ is a positive integer exactly when $s \leq 7$, since $40-5\cdot 8 = 0$. So a collection of bills is the same as a pair of positive integers (w, z) with $w + z = s$ for some $s \leq 7$; the value of x is then determined. For a fixed s, the number of positive integer solutions of $w + z = s$ is $\binom{s-1}{1} = s - 1$. Adding over $s = 2, 3, \ldots, 7$ gives

$$1 + 2 + 3 + 4 + 5 + 6 = 21.$$

Answer: (B) 21. □

7. Four people are to be seated at a round table with 12 equally spaced chairs so that no two of them sit in neighboring chairs. Two seatings are considered the same if one is a rotation of the other. How many different seatings are possible?

Strategy. As in the count of circular permutations, turn each seating so that one particular person sits in a fixed chair. That person's two neighboring chairs stay empty, and the other chairs form a row in which the remaining three people need chairs with no two adjacent.

Solution. Number the chairs 1 to 12 around the table, and call one of the four people P. The 12 rotations of a seating put P in 12 different chairs, so exactly one of them puts P in chair 1. Hence the seatings up to rotation correspond one to one to the seatings with P in chair 1. Count these.

With P in chair 1, chairs 2 and 12 must be empty, so the other three people sit in chairs 3 to 11. These 9 chairs form a row whose end chairs, 3 and 11, are not neighbors. No two of the three people sit in neighboring chairs exactly when their three chairs include no two

adjacent chairs of this row. By the count of non-adjacent selections, the three chairs can be chosen in

$$\binom{9-3+1}{3} = \binom{7}{3} = 35$$

ways, and the three people can sit in the chosen chairs in $3! = 6$ orders. By the Multiplication Principle, the number of seatings is $35 \cdot 6 = 210$.

Answer: 210. □

8. (2010 AMC 10B, Problem #22) Seven distinct pieces of candy are to be distributed among three bags. The red bag and the blue bag must each receive at least one piece of candy; the white bag may remain empty. How many arrangements are possible?

(A) 1930 **(B)** 1931 **(C)** 1932 **(D)** 1933 **(E)** 1934

Strategy. The pieces are distinct and the bags are labeled, so each piece independently chooses a bag. Use inclusion-exclusion to remove the distributions that leave the red bag or the blue bag empty.

Solution. Each of the 7 pieces can go into any of the 3 bags, so there are $3^7 = 2187$ distributions in all. Let R be the set of distributions in which the red bag is empty and B the set in which the blue bag is empty. In R, every piece goes into the blue or the white bag, so $|R| = 2^7 = 128$; similarly $|B| = 128$. In $R \cap B$, every piece goes into the white bag, so $|R \cap B| = 1$. By inclusion-exclusion,

$$|R \cup B| = 128 + 128 - 1 = 255.$$

The allowed distributions are all the others, so there are $2187 - 255 = 1932$.

Answer: (C) 1932. □

9. (1990 AIME, Problem #8) In a shooting match, eight clay targets are arranged in two hanging columns of three targets each and one column of two targets. A marksman is to break all the targets according to the following rules:
(1) The marksman first chooses a column from which a target is to be broken.
(2) The marksman must then break the lowest remaining target in the chosen column.
If the rules are followed, in how many different orders can the eight targets be broken?

Strategy. Rule (2) fixes the order in which the targets of each column are broken, so an order is determined by which of the eight shots go to each column. That is a division of the eight shots into three labeled groups of sizes 3, 3 and 2.

Solution. Call the two columns of three targets A and B and the column of two targets C, and number the shots 1 to 8 in the order they are fired. An order of breaking that follows the rules determines, for each column, the set of shots that break a target in that column. Every target is broken by exactly one shot, so these three sets divide the 8 shots into three labeled groups A, B and C of sizes 3, 3 and 2.

Conversely, the division determines the order. By rule (2), the targets of a column are broken from the bottom up: the earliest shot in the group of a column breaks its lowest target, the next shot in that group breaks the target above it, and so on. So the division tells which target each shot breaks. Every division does come from an order, because each column is chosen exactly as many times as it has targets, so a chosen column always has a target left.

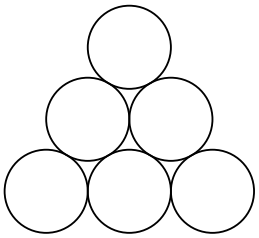
Hence the orders correspond one to one to the divisions, and by the multinomial coefficient their number is

$$\binom{8}{3,3,2} = \frac{8!}{3!\,3!\,2!} = \frac{40{,}320}{72} = 560.$$

Equivalently, listing the column of each shot in turn gives an arrangement of A, A, A, B, B, B, C, C in a row, and the same multinomial coefficient counts these arrangements.

Answer: 560. □

10. (2017 AMC 10B, Problem #18) In the figure below, 3 of the 6 disks are to be painted blue, 2 are to be painted red, and 1 is to be painted green. Two paintings that can be obtained from one another by a rotation or a reflection of the entire figure are considered the same. How many different paintings are possible?



(A) 6 **(B)** 8 **(C)** 9 **(D)** 12 **(E)** 15

Strategy. The green disk is the only disk of its color. Turn the figure so that the green disk is the top disk or the middle disk of the bottom row. Then only a reflection can keep it in place, so pair each painting with its mirror image, as for bracelets, and count once each painting that is its own mirror image.

Solution. Call the three disks at the corners of the triangle the corner disks and the other three the side disks. A rotation or a reflection of the figure takes corner disks to corner disks and side disks to side disks. So a painting with the green disk at a corner is never the same as one with the green disk at a side, and the two cases can be counted separately.

Green disk at a corner. One of the rotations by 0°, 120° and 240° moves the green disk to the top, so it is enough to count the paintings with the green disk at the top. If a rotation or reflection takes one such painting to another, it takes the green disk to the green disk, so it keeps the top disk in place. The only such motions are doing nothing and the reflection M in the vertical line through the top disk. So two paintings with the green disk at the top are the same exactly when they are equal or one is the mirror image of the other under M.

The reflection M keeps the middle disk of the bottom row in place, swaps the two disks of the middle row, and swaps the two bottom corner disks. The other five disks are painted with 3 blue and 2 red in $\binom{5}{2} = 10$ ways, one for each choice of the two red disks. A painting is its own mirror image exactly when the two disks of each swapped pair have the same color, that is, when the two red disks are the two disks of the middle row or the two bottom corner disks. So 2 of the 10 paintings are their own mirror images. The other 8 split into 4 pairs, each made of a painting and its mirror image, and each pair is one painting up to rotation and reflection. This case gives $2 + 4 = 6$ paintings.

Green disk at a side. In the same way, turn the figure so that the green disk is the middle disk of the bottom row. The motions that keep this disk in place are again doing nothing and M, which now keeps the top disk in place and swaps the same two pairs. So this case also gives 6 paintings.

The number of different paintings is $6 + 6 = 12$.
Answer: (D) 12. □

Advanced.

11. (2026 AIME I, Problem #2) Find the number of positive integer palindromes written in base 10, with no zero digits, and whose digits add up to 13. For example, 42124 has these properties. Recall that a palindrome is a number whose representation reads the same from left to right as from right to left.

Strategy. A palindrome is determined by the first half of its digits, together with the middle digit when the number of digits is odd. Since 13 is odd, there must be a middle digit, and the first half is an integer composition of a small number.

Solution. Consider a palindrome with no zero digits and digit sum 13, with L digits. Its ith digit from the left equals its ith digit from the right. If L were even, the digits would split into pairs of equal digits, and the digit sum would be even. So $L = 2m + 1$ is odd. Let c be the middle digit and $d_1, d_2, \ldots, d_m$ the first m digits. The last m digits are the same digits in the reverse order, so

$$2(d_1 + d_2 + \cdots + d_m) + c = 13.$$

Hence c is odd and $d_1 + d_2 + \cdots + d_m = s$, where $s = (13 - c)/2$. The digit c is one of $1, 3, 5, 7, 9$, so s is one of $6, 5, 4, 3, 2$, and in particular $m \geq 1$.

For a fixed c, the palindromes with middle digit c correspond one to one to the lists $(d_1, d_2, \ldots, d_m)$ of positive integers with sum s, of any length m. Each term of such a list is at most $s \leq 6$, so it is a nonzero digit. These lists are the integer compositions of s. By the count of positive integer solutions, s has $\binom{s-1}{m-1}$ compositions into m parts, so by the sum of a row of Pascal's triangle it has

$$\binom{s-1}{0} + \binom{s-1}{1} + \cdots + \binom{s-1}{s-1} = 2^{s-1}$$

compositions in all. Adding over $s = 6, 5, 4, 3, 2$, the number of palindromes is

$$2^5 + 2^4 + 2^3 + 2^2 + 2^1 = 32 + 16 + 8 + 4 + 2 = 62.$$

Answer: 62. □

12. (2024 AMC 10B, Problem #22) A group of 16 people will be partitioned into 4 indistinguishable 4-person committees. Each committee will have one chairperson and one secretary. The number of different ways to make these assignments can be written as $3^r M$, where r and M are positive integers and M is not divisible by 3. What is r?
(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

Strategy. Count the assignments with the rule for unlabeled groups, then count the factors of 3 in the result.

Solution. First split the 16 people into 4 labeled committees of 4, which can be done in $16!/(4!)^4$ ways. The committees are indistinguishable and have equal sizes, so divide by 4!, the number of ways to label them. In each committee the chairperson can be chosen in 4 ways and then the secretary in 3 ways, which gives 12 choices per committee and 12^4 in all. So the number of assignments is

$$N = \frac{16!}{(4!)^4 \cdot 4!} \cdot 12^4 = \frac{16! \cdot 12^4}{(4!)^5}.$$

Now count factors of 3. Among $1, 2, \ldots, 16$, the multiples of 3 are 3, 6, 9, 12, and 15, and $9 = 3^2$ contributes one more factor, so 16! has 6 factors of 3. Since $12^4 = 2^8 \cdot 3^4$, it has 4 factors of 3, and since $4! = 24 = 2^3 \cdot 3$, the product $(4!)^5$ has 5. Therefore N has exactly $6 + 4 - 5 = 5$ factors of 3, so $r = 5$.

Answer: (A) 5. □

13. (2025 AIME I, Problem #3) The 9 members of a baseball team went to an ice-cream parlor after their game. Each player had a single scoop cone of chocolate, vanilla, or strawberry ice cream. At least one player chose each flavor, and the number of players who chose chocolate was greater than the number of players who chose vanilla, which was greater than the number of players who chose strawberry. Let N be the number of different assignments of flavors to players that meet these conditions. Find the remainder when N is divided by 1000.

Strategy. When the numbers of players who chose each flavor are fixed, an assignment is a division of the 9 players into three labeled groups of those sizes. Find the possible sizes, and add the multinomial coefficients.

Solution. Let c, v and s be the numbers of players who chose chocolate, vanilla and strawberry. Then $c + v + s = 9$ and $c > v > s \geq 1$. Since $v \geq s + 1$ and $c \geq s + 2$,

$$9 = c + v + s \geq 3s + 3,$$

so s is 1 or 2. If $s = 1$, then $c + v = 8$ with $c > v \geq 2$, so (c, v) is $(6, 2)$ or $(5, 3)$. If $s = 2$, then $c + v = 7$ with $c > v \geq 3$, so $(c, v) = (4, 3)$. The possible triples (c, v, s) are $(6, 2, 1)$, $(5, 3, 1)$ and $(4, 3, 2)$.

For fixed sizes, an assignment is a division of the 9 players into the chocolate, vanilla and strawberry groups, which are labeled groups of sizes c, v and s. By the multinomial coefficient, the three triples give

$$\frac{9!}{6!\,2!\,1!} = 252, \qquad \frac{9!}{5!\,3!\,1!} = 504, \qquad \frac{9!}{4!\,3!\,2!} = 1260$$

assignments. By the Addition Principle, $N = 252 + 504 + 1260 = 2016$, and the remainder when N is divided by 1000 is 16.

Answer: 16. □

14. (1996 AIME, Problem #7) Two of the squares of a 7×7 checkerboard are painted yellow, and the rest are painted green. Two color schemes are equivalent if one can be obtained from the other by applying a rotation in the plane of the board. How many inequivalent color schemes are possible?

Strategy. Count the color schemes on a board held in place, and group each scheme with its rotations, as in the count of circular permutations. Here a half turn can leave a scheme unchanged, so the groups do not all have the same size.

Solution. With the board held in place, a color scheme is a choice of the 2 yellow squares among the 49 squares, so there are $\binom{49}{2} = 1176$ schemes. The board can be rotated about its center by 0°, 90°, 180° or 270°. Put each scheme in a group with its rotations. Two schemes are equivalent exactly when they are in the same group, so the answer is the number of groups.

A rotation by 90°, 180° or 270° leaves only the center square in place. So if such a rotation leaves a scheme unchanged, neither yellow square is the center square, since the other one would have to stay in place, and the rotation swaps the two yellow squares. A quarter turn, by 90° or 270°, cannot swap two squares: doing it twice is a half turn, which

would then leave both squares in place. A half turn swaps the two yellow squares exactly when they are symmetric about the center of the board, and the 48 squares other than the center form 24 such pairs. Hence no scheme is unchanged by a quarter turn, and exactly 24 schemes, the symmetric ones, are unchanged by a half turn.

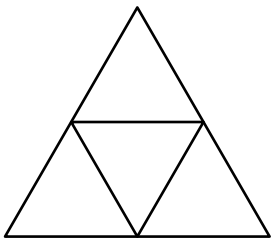
For each of the 24 symmetric schemes, the rotations by $0°$ and $180°$ give the scheme itself, and the rotations by $90°$ and $270°$ give one other symmetric scheme. So these schemes form groups of 2, and there are $24/2 = 12$ such groups.

For each of the other $1176 - 24 = 1152$ schemes, the four rotations give four different schemes: if two of them were equal, the rotation by the difference of their angles, $90°$, $180°$ or $270°$, would leave the scheme unchanged. So these schemes form groups of 4, and there are $1152/4 = 288$ such groups.

The number of inequivalent color schemes is $12 + 288 = 300$.

Answer: 300. □

15. (2006 AIME II, Problem #8) There is an unlimited supply of congruent equilateral triangles made of colored paper. Each triangle is a solid color with the same color on both sides of the paper. A large equilateral triangle is constructed from four of these paper triangles as shown. Two large triangles are considered distinguishable if it is not possible to place one on the other, using translations, rotations, and/or reflections, so that their corresponding small triangles are of the same color. Given that there are six different colors of triangles from which to choose, how many distinguishable large equilateral triangles can be constructed?



Strategy. Every placement of one large triangle on another puts the center triangle on the center triangle, and the placements can match the corner triangles in any order. So a large triangle is described by the color of its center triangle and the colors of its corner triangles without regard to order, which is a combination with repetition.

Solution. Call the small triangle in the middle the center triangle and the other three the corner triangles. A placement of one large triangle on another takes the midpoints of the sides of the first to the midpoints of the sides of the second, so it puts the center triangle on the center triangle and the corner triangles on the corner triangles. The rotations of the large triangle by $0°$, $120°$ and $240°$ move the corner triangles around in a cycle, and the reflection in a line through one corner and the midpoint of the opposite side keeps that corner triangle in place and swaps the other two. Together these six motions match up the corner triangles in all $3! = 6$ possible ways. A reflection turns the paper triangles over, which does not change their colors.

Hence one large triangle can be placed on another with all colors matching exactly when their center triangles have the same color and their corner triangles use the same colors, each the same number of times. Therefore the distinguishable large triangles correspond to the choices of a color for the center triangle, in 6 ways, and of the colors of the three corner triangles without regard to order. The latter is a choice of 3 items from 6 types, where a type may be chosen more than once and order does not matter. By the count of

combinations with repetition with $n = 6$ and $k = 3$, there are

$$\binom{6+3-1}{3} = \binom{8}{3} = 56$$

such choices. By the Multiplication Principle, the number of distinguishable large triangles is $6 \cdot 56 = 336$.

Answer: 336. □

2. Answer Key

Problem	Letter	Value
1	B	18
2		1540
3		768
4	C	3
5		144
6	B	21
7		210
8	C	1932
9		560
10	D	12
11		62
12	A	5
13		16
14		300
15		336