COMBINATIONS AND THE BINOMIAL THEOREM FOR AMC 10

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1. Combinations

This section counts selections in which the order of the chosen objects does not matter.

1.1. Choosing a Subset.

A club of 10 members can choose a president, a vice president and a secretary in $P(10,3) = 10 \cdot 9 \cdot 8 = 720$ ways, because the three offices are different positions. If the club instead chooses a committee of 3 members with no offices, then choosing Ann, Ben and Cal gives the same committee as choosing Cal, Ann and Ben. Only the set of chosen members matters.

Definition 1.1.1. A **subset** of a set is a set formed from some of its elements, possibly none or all of them. A **combination** of k objects from n distinct objects is a choice of k of the objects in which the order does not matter, so the chosen objects form a k-element subset. The number of k-element subsets of an n-element set is written $\binom{n}{k}$ and read "n **choose** k". The numbers $\binom{n}{k}$ are also called **binomial coefficients**, for a reason given in Section 4.

Theorem 1.1.2. *For $0 \le k \le n$,*

$$\binom{n}{k} = \frac{P(n,k)}{k!} = \frac{n(n-1)\cdots(n-k+1)}{k!} = \frac{n!}{k!\,(n-k)!}.$$

Proof. Count the arrangements of k of the n objects in a row in two ways. By the formula for permutations, there are $P(n,k)$ of them. Alternatively, first choose the set of k objects, in $\binom{n}{k}$ ways, and then arrange the chosen objects in a row, in $k!$ ways. Different choices give different arrangements, and every arrangement arises in this way, so by the Multiplication Principle

$$P(n,k) = \binom{n}{k} \cdot k!.$$

Dividing by $k!$ and using $P(n,k) = n!/(n-k)!$ gives the formula. □

In the club, each committee of 3 appears $3! = 6$ times among the 720 ordered choices, once for each way to give its three members the three offices, so there are $720/6 = 120$ committees. In practice, compute $\binom{n}{k}$ as a product of k decreasing factors divided by $k!$:

$$\binom{10}{3} = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = 120.$$

The following values come up often:

$$\binom{n}{0} = \binom{n}{n} = 1, \qquad \binom{n}{1} = n, \qquad \binom{n}{2} = \frac{n(n-1)}{2}.$$

The first says that a set has one subset with no elements and one subset containing all of its elements; the formula gives it because $0! = 1$.

Remark 1.1.3. A k-element subset of $\{1, 2, \ldots, n\}$ can be recorded as a row of n letters: write Y in position i if i is chosen and N if it is not. For example, the subset $\{2,3,5\}$ of $\{1,2,3,4,5\}$ is recorded as NYYNY. Each subset gives exactly one row of k Y's and $n-k$ N's, and each such row comes from exactly one subset. By the formula for arrangements of a multiset, there are $n!/(k!\,(n-k)!)$ such rows, which is $\binom{n}{k}$ again. So $\binom{n}{k}$ also counts the arrangements in a row of k identical objects of one kind and $n-k$ identical objects of another kind.

Use $\binom{n}{k}$ when the chosen objects all play the same role, and $P(n,k)$ when they fill different positions.

Examples.

Problem 1.1.4. A coach must choose 4 of the 10 players on her team.

(a) In how many ways can she choose 4 players to run the four legs of a relay race, first, second, third and fourth?
(b) In how many ways can she choose 4 players to travel to a tournament?

Strategy. In (a) the four legs are different positions, so the order of the chosen players matters. In (b) the traveling group is a set of players.

Solution. (a) Any of the 10 players can run the first leg, any of the other 9 the second leg, any of the remaining 8 the third leg, and any of the remaining 7 the fourth leg. So there are

$$P(10, 4) = 10 \cdot 9 \cdot 8 \cdot 7 = 5040$$

ways.

(b) Each group of 4 players appears $4! = 24$ times among the 5040 ordered choices of part (a), once for each way to assign its members to the four legs. So there are $5040/24 = 210$ groups, which is

$$\binom{10}{4} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1} = 210.$$

Answer: (a) 5040 and (b) 210. □

Problem 1.1.5 (1989 AHSME, Problem #9)**.** Mr. and Mrs. Zeta want to name their baby Zeta so that its monogram (first, middle, and last initials) will be in alphabetical order with no letters repeated. How many such monograms are possible?

(A) 276 **(B)** 300 **(C)** 552 **(D)** 600 **(E)** 15600

Strategy. Only the first and middle initials can vary, and alphabetical order fixes their order. So count sets of two letters, not ordered pairs.

Solution. The baby's last name is Zeta, so the last initial is Z. The first and middle initials are two different letters that come before Z in the alphabet, so they are two of the 25 letters A through Y. Once these two letters are chosen, alphabetical order decides which of them is the first initial. So each monogram comes from exactly one 2-element subset of the letters A through Y, and each such subset gives exactly one monogram. The number of monograms is

$$\binom{25}{2} = \frac{25 \cdot 24}{2} = 300.$$

Counting ordered pairs of letters instead, $25 \cdot 24 = 600$, would count each monogram twice, once for each order of its two letters.

Answer: (B) 300. □

1.2. **Symmetry.**

Theorem 1.2.1 (Symmetry)**.** *For* $0 \le k \le n$,

$$\binom{n}{k} = \binom{n}{n-k}.$$

Proof. Choosing the k elements of a subset is the same as choosing the $n - k$ elements left out. Each k-element subset leaves out exactly one $(n - k)$-element subset, and each $(n - k)$-element subset is left out by exactly one k-element subset, so the two numbers are equal. The formula shows the same thing: exchanging k and $n - k$ does not change $n!/(k!\,(n - k)!)$. □

When k is more than half of n, compute $\binom{n}{n-k}$ instead, which has fewer factors:

$$\binom{20}{17} = \binom{20}{3} = \frac{20 \cdot 19 \cdot 18}{3 \cdot 2 \cdot 1} = 1140.$$

Example.

Problem 1.2.2 (2013 AMC 10A, Problem #11)**.** A student council must select a two-person welcoming committee and a three-person planning committee from among its members. There are exactly 10 ways to select a two-person team for the welcoming committee. It is possible for students to serve on both committees. In how many different ways can a three-person planning committee be selected?

(A) 10 **(B)** 12 **(C)** 15 **(D)** 18 **(E)** 25

Strategy. Find the number of members from $\binom{n}{2} = 10$, and then use symmetry.

Solution. Let the council have n members. The number of two-person teams is

$$\binom{n}{2} = \frac{n(n-1)}{2} = 10,$$

so $n(n-1) = 20$. The product $n(n-1)$ increases with n, and $5 \cdot 4 = 20$, so $n = 5$. Students may serve on both committees, so the planning committee is any 3 of the 5 members. Choosing the 3 members of the planning committee is the same as choosing the 2 members left off it, so the number of planning committees is

$$\binom{5}{3} = \binom{5}{2} = 10.$$

Answer: (A) 10. □

2. Counting with Combinations

This section combines binomial coefficients with the Addition and Multiplication Principles.

2.1. **Committees with Conditions.** To choose a people from a group of m people and b people from a separate group of p people, choose the two parts one after the other. The second part can be chosen in $\binom{p}{b}$ ways whatever the first part is, so by the Multiplication Principle there are

$$\binom{m}{a}\binom{p}{b}$$

ways.

If a particular person must be on a k-member committee chosen from n people, put that person on the committee first and choose the other $k-1$ members from the remaining $n-1$ people, in $\binom{n-1}{k-1}$ ways. If the person must be left off, all k members come from the other $n-1$ people, in $\binom{n-1}{k}$ ways. As with permutations, make the restricted choices first.

Example.

Problem 2.1.1. A club has 6 seniors and 8 juniors. One of the seniors is named Maya. A committee of 5 club members is to be chosen.

(a) How many committees contain exactly 2 seniors?
(b) How many of the committees in part (a) include Maya?

Strategy. Choose the seniors and the juniors separately. In (b), put Maya on the committee first.

Solution. (a) The 2 seniors can be chosen from the 6 seniors in $\binom{6}{2} = 15$ ways, and the other 3 members are juniors, chosen from the 8 juniors in $\binom{8}{3} = 56$ ways. By the Multiplication Principle there are $15 \cdot 56 = 840$ committees.

(b) Put Maya on the committee first. The other senior is one of the remaining 5 seniors, so it can be chosen in $\binom{5}{1} = 5$ ways, and the juniors can be chosen in 56 ways as before. So $5 \cdot 56 = 280$ of the committees include Maya.

The other $840 - 280 = 560$ committees of part (a) leave Maya off. Indeed, choosing both seniors from the other 5 gives $\binom{5}{2} \cdot 56 = 10 \cdot 56 = 560$ committees. The split $15 = 5 + 10$ of the choices of the seniors appears again in Section 3.1 as Pascal's identity.

Answer: (a) 840 and (b) 280. □

2.2. **Casework with Combinations.** When a condition depends on how many of the chosen objects are of each type, such as "at least two women" or "the sum is even", list the possible numbers of objects of each type. Count each case as a product of binomial coefficients, and add the cases by the Addition Principle. The cases must not overlap, and together they must include every possibility. A condition such as "at least" or "at most" becomes a short list of cases.

Examples.

Problem 2.2.1 (1996 AJHSME, Problem #21)**.** How many subsets containing three different numbers can be selected from the set $\{89, 95, 99, 132, 166, 173\}$ so that the sum of the three numbers is even?

(A) 6 **(B)** 8 **(C)** 10 **(D)** 12 **(E)** 24

Strategy. Only whether each number is odd or even matters. Use cases on how many odd numbers are chosen.

Solution. Whether a sum of three integers is even depends only on how many of the three are odd: the sum is even exactly when none or two of them are odd. In the given set, only 132 and 166 are even, and the other four numbers are odd. Three numbers chosen from the set cannot all be even, so a subset with an even sum has exactly two odd numbers. Such a subset consists of a pair of the four odd numbers, chosen in $\binom{4}{2} = 6$ ways, and one of the two even numbers, chosen in 2 ways. By the Multiplication Principle, $6 \cdot 2 = 12$ subsets have an even sum.

Answer: (D) 12. □

Problem 2.2.2 (1989 AHSME, Problem #22)**.** A child has a set of 96 distinct blocks. Each block is one of 2 materials (*plastic, wood*), 3 sizes (*small, medium, large*), 4 colors (*blue, green, red, yellow*), and 4 shapes (*circle, hexagon, square, triangle*). How many blocks in the set are different from the "plastic medium red circle" in exactly two ways? (The "wood medium red square" is such a block.)

(A) 29 **(B)** 39 **(C)** 48 **(D)** 56 **(E)** 62

Strategy. Choose which two of the four features differ. The number of blocks depends on which two they are, so add over the $\binom{4}{2} = 6$ choices.

Solution. A block differs from the plastic medium red circle in exactly two ways when exactly two of its four features differ and the other two are the same. For each feature, the number of values other than the given one is 1 for the material, 2 for the size, 3 for the color and 3 for the shape. Choose the two features that differ, in one of $\binom{4}{2} = 6$ ways. The number of blocks is then the product of the numbers of other values of these two features.

Features that differ	Blocks
material and size	$1 \cdot 2 = 2$
material and color	$1 \cdot 3 = 3$
material and shape	$1 \cdot 3 = 3$
size and color	$2 \cdot 3 = 6$
size and shape	$2 \cdot 3 = 6$
color and shape	$3 \cdot 3 = 9$

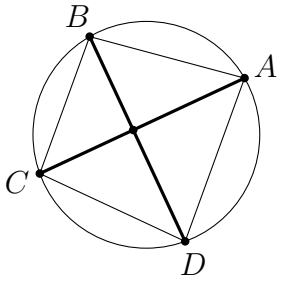
The six cases do not overlap, so by the Addition Principle there are $2 + 3 + 3 + 6 + 6 + 9 = 29$ blocks.

Answer: (A) 29. □

2.3. **Objects Determined by a Subset.** Some objects are not sets themselves, but each one is determined by a set. Suppose that each object to be counted gives a set of k elements chosen from a fixed set of n elements, that different objects give different sets, and that every k-element set comes from some object. Then there are exactly $\binom{n}{k}$ objects. Both conditions must be checked: different objects give different sets, and every set gives an object.

A positive integer whose digits strictly increase from left to right is determined by its set of digits: writing the digits of the set in increasing order gives the integer back. The digit 0 cannot appear, because it would have to be the first digit. So the integers with k strictly increasing digits correspond to the k-element subsets of $\{1, 2, \ldots, 9\}$, and there are $\binom{9}{k}$ of them. For example, $\binom{9}{3} = 84$ three-digit integers have strictly increasing digits.

Two chords of a circle with four different endpoints cross inside the circle exactly when their endpoints alternate around the circle, as the endpoints of chords AC and BD do in the figure. Two chords with a common endpoint meet only at that endpoint, on the circle. So if n points on a circle are joined by all the chords, and no three chords pass through one point inside the circle, then each set of 4 of the points gives exactly one crossing point, where the diagonals of the quadrilateral they form cross. Conversely, a crossing point inside the circle lies on exactly two chords, since no three chords pass through it, and these two chords have 4 different endpoints, so the crossing point comes from exactly one set of 4 points. Hence there are $\binom{n}{4}$ crossing points inside the circle.



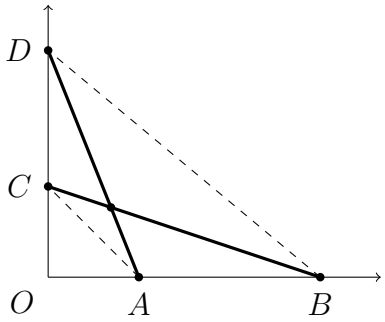
Example.

Problem 2.3.1 (1992 AHSME, Problem #22)**.** Ten points are selected on the positive x-axis, X^+, and five points are selected on the positive y-axis, Y^+. The fifty segments connecting the ten selected points on X^+ to the five selected points on Y^+ are drawn. What is the maximum possible number of points of intersection of these fifty segments that could lie in the interior of the first quadrant?

(A) 250 **(B)** 450 **(C)** 500 **(D)** 1250 **(E)** 2500

Strategy. Decide which pairs of segments can cross inside the quadrant. Then describe each crossing by the endpoints of its two segments.

Solution. Two segments with a common endpoint meet only at that endpoint, which lies on an axis. So every intersection point inside the quadrant lies on two segments with four different endpoints, two on X^+ and two on Y^+. Call them A and B on X^+, with A closer to the origin O, and C and D on Y^+, with C closer to O.



The segments AC and BD (dashed) do not meet: A lies between O and B, and C lies between O and D, so segment AC lies on the same side of line BD as O. The segments AD and BC cross: A lies between O and B while D lies beyond C, so A and D are on opposite sides of line BC, and in the same way B and C are on opposite sides of line AD. So each choice of 2 points on X^+ and 2 points on Y^+ gives exactly one intersection point inside the quadrant, and every such point comes from one of these choices. There are

$$\binom{10}{2}\binom{5}{2} = 45 \cdot 10 = 450$$

choices, so there are at most 450 intersection points. Different choices give the same point only when three or more segments pass through it, and this can be avoided. Place the five points on Y^+ first, and then the points on X^+ one at a time. The segments from a new point meet one another only at that point, on X^+. For each intersection point already drawn and each point on Y^+, the line through these two points meets X^+ at most once, and the new point avoids these finitely many positions. Then no new segment passes through an earlier intersection point, so at no stage do three segments pass through one point, and the 450 choices give 450 different intersection points.

Answer: (B) 450. □

3. Pascal's Triangle

Writing the binomial coefficients in rows shows several patterns that are used throughout counting.

3.1. **Pascal's Identity.**

Definition 3.1.1. Pascal's triangle is the triangular array whose row n, for $n = 0, 1, 2, \ldots$, lists the numbers $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$. Rows 0 to 8 are shown below, with row 0 at the top.

$$\begin{array}{ccccccccccccccccc}
 & & & & & & & 1 & & & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & & & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 & & \\
1 & & 8 & & 28 & & 56 & & 70 & & 56 & & 28 & & 8 & & 1
\end{array}$$

Each row begins and ends with 1. Each row reads the same from right to left as from left to right, by symmetry. Each entry inside a row is the sum of the two entries above it; for example, $10 = 4 + 6$ in row 5.

Theorem 3.1.2 (Pascal's Identity)**.** *For* $1 \le k \le n-1$,

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Proof. Count the k-member committees chosen from n people, one of whom is Ann. A committee that includes Ann consists of Ann and $k-1$ of the other $n-1$ people, so there are $\binom{n-1}{k-1}$ of these. A committee that leaves Ann off consists of k of the other $n-1$ people, so there are $\binom{n-1}{k}$ of these. Every committee is in exactly one of the two groups, so the two numbers add up to $\binom{n}{k}$. □

Problem 2.1.1 showed the case $n = 6$, $k = 2$: $\binom{6}{2} = \binom{5}{1} + \binom{5}{2}$, that is, $15 = 5 + 10$. By the identity, each row of the triangle can be built from the row above it, and a sum of two neighboring entries of a row is a single entry of the next row.

Example.

Problem 3.1.3. Write $\binom{9}{3} + \binom{9}{4} + \binom{10}{5}$ as a single binomial coefficient, and find its value.

Strategy. Apply Pascal's identity to the first two terms, and then again.

Solution. By Pascal's identity with $n = 10$ and $k = 4$, $\binom{9}{3} + \binom{9}{4} = \binom{10}{4}$. Then, by Pascal's identity with $n = 11$ and $k = 5$,

$$\binom{10}{4} + \binom{10}{5} = \binom{11}{5} = \frac{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 462.$$

As a check, $\binom{9}{3} + \binom{9}{4} + \binom{10}{5} = 84 + 126 + 252 = 462$.

Answer: $\binom{11}{5} = 462$. □

3.2. **Neighboring Entries in a Row.**

Proposition 3.2.1. For $0 \le k \le n-1$,

$$\binom{n}{k+1} = \binom{n}{k} \cdot \frac{n-k}{k+1}.$$

Proof. Since $(k+1)! = (k+1) \cdot k!$ and $(n-k)! = (n-k) \cdot (n-k-1)!$,

$$\binom{n}{k+1} = \frac{n!}{(k+1)!\,(n-k-1)!} = \frac{n!}{k!\,(n-k)!} \cdot \frac{n-k}{k+1} = \binom{n}{k} \cdot \frac{n-k}{k+1}. \qquad \square$$

The factor $\frac{n-k}{k+1}$ is greater than 1 when $k+1 < n-k$, that is, in the first half of the row; it equals 1 when $k+1 = n-k$, and it is less than 1 after that. So the entries of a row increase up to the middle and then decrease. The largest entry of row n is its middle entry when n is even; when n is odd, the two middle entries are equal and largest. For example, row 8 starts with 1, and the factors $\frac{8}{1}$, $\frac{7}{2}$, $\frac{6}{3}$, $\frac{5}{4}$ give the entries 8, 28, 56, 70; the next factor, $\frac{4}{5}$, is less than 1.

Example.

Problem 3.2.2 (2024 AMC 10B, Problem #12)**.** A group of 100 students from different countries meet at a mathematics competition. Each student speaks the same number of languages, and, for every pair of students A and B, student A speaks some language that student B does not speak, and student B speaks some language that student A does not speak. What is the least possible total number of languages spoken by all the students?

(A) 9 **(B)** 10 **(C)** 12 **(D)** 51 **(E)** 100

Strategy. The condition says that the students' sets of languages are all different. Count how many different sets of the same size a given number of languages allows.

Solution. Suppose each student speaks k languages and n languages are spoken in all. Two different sets with the same number of elements each contain an element that the other does not, and two equal sets do not. So the condition says exactly that the 100 students speak 100 different sets of k languages, chosen from the n languages. Hence $\binom{n}{k} \ge 100$.

As shown after Proposition 3.2.1, the largest entry of a row of Pascal's triangle is in the middle. The middle entries of rows 0 to 8 are $1, 1, 2, 3, 6, 10, 20, 35, 70$, all less than 100, so $n \le 8$ is impossible. With $n = 9$ and $k = 4$ there are $\binom{9}{4} = 126$ sets of 4 languages. Give the students 100 different sets of 4 of the languages $1, 2, \ldots, 9$, chosen so that every language is used, for example by including $\{1,2,3,4\}$, $\{5,6,7,8\}$ and $\{6,7,8,9\}$ among them. Then each student speaks 4 languages, the condition holds for every pair of students, and exactly 9 languages are spoken. So the least possible number is 9.

Answer: (A) 9. $\square$

4. The Binomial Theorem

Binomial coefficients get their name from the expansion of a power of a **binomial**, a sum of two terms such as $x + y$.

4.1. **Expanding a Power of a Binomial.** Multiplying out the first few powers of $x + y$ gives

$$\begin{aligned}(x+y)^2 &= x^2 + 2xy + y^2,\\ (x+y)^3 &= x^3 + 3x^2y + 3xy^2 + y^3,\\ (x+y)^4 &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.\end{aligned}$$

The coefficients are rows 2, 3 and 4 of Pascal's triangle.

Theorem 4.1.1 (Binomial Theorem)**.** *For every positive integer n,*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k}y^k = x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}xy^{n-1} + y^n.$$

Proof. The power $(x+y)^n$ is a product of n factors, each equal to $x+y$. Multiplying out gives a sum of products, one for each way to pick either x or y from every factor. Such a product equals $x^{n-k}y^k$ exactly when y is picked from k of the n factors and x from the others, and there are $\binom{n}{k}$ ways to choose these k factors. So after like terms are collected, the coefficient of $x^{n-k}y^k$ is $\binom{n}{k}$. □

This is why the numbers $\binom{n}{k}$ are called binomial coefficients. In the expansion of $(x+y)^n$, the term containing y^k is $\binom{n}{k}x^{n-k}y^k$. Written in decreasing powers of x, it is the $(k+1)$st term, and the expansion has $n+1$ terms.

The theorem holds with any numbers or expressions in place of x and y. To expand $(2x-1)^6$, for example, use $2x$ in place of x and -1 in place of y; the signs of the terms then alternate. To find a single coefficient, write the term containing y^k, simplify its power of the variable, and solve for k.

Examples.

Problem 4.1.2. Find the coefficient of x^3 in the expansion of $(2x-1)^6$.

Strategy. Use the binomial theorem with $2x$ and -1, and find the value of k that gives x^3.

Solution. By the binomial theorem with $2x$ in place of x and -1 in place of y, the term containing y^k becomes

$$\binom{6}{k}(2x)^{6-k}(-1)^k = \binom{6}{k}2^{6-k}(-1)^k\,x^{6-k}.$$

Its power of x is x^3 when $6-k=3$, that is, when $k=3$. The coefficient is

$$\binom{6}{3}\cdot 2^3\cdot(-1)^3 = 20\cdot 8\cdot(-1) = -160.$$

Answer: -160. □

Problem 4.1.3 (1960 AHSME, Problem #20)**.** The coefficient of x^7 in the expansion of

$$\left(\frac{x^2}{2} - \frac{2}{x}\right)^8$$

is:

(A) 56 **(B)** -56 **(C)** 14 **(D)** -14 **(E)** 0

Strategy. Write the term containing $(-2/x)^k$, and find which value of k gives x^7.

Solution. By the binomial theorem with $x^2/2$ in place of x and $-2/x$ in place of y, the term containing $(-2/x)^k$ is

$$\binom{8}{k}\left(\frac{x^2}{2}\right)^{8-k}\left(-\frac{2}{x}\right)^k = \binom{8}{k}(-1)^k\cdot\frac{2^k}{2^{8-k}}\cdot\frac{x^{16-2k}}{x^k} = \binom{8}{k}(-1)^k\,2^{2k-8}\,x^{16-3k}.$$

It contains x^7 when $16-3k=7$, that is, when $k=3$. The coefficient is

$$\binom{8}{3}(-1)^3\,2^{-2} = -\frac{56}{4} = -14.$$

Answer: (D) -14. □

4.2. **Estimates and Last Digits.** When ny is small compared with x, for positive x and y, the terms of the expansion of $(x+y)^n$ shrink quickly, and the first few terms give an accurate value. Indeed, by Proposition 3.2.1, the term containing y^{k+1} is $\frac{n-k}{k+1}\cdot\frac{y}{x}$ times the term containing y^k, and this factor is at most $\frac{ny}{x}$. For example,

$$(1.01)^5 = (1+0.01)^5 \approx 1 + 5(0.01) + 10(0.01)^2 = 1.051,$$

while the exact value is 1.0510100501.

When x is an integer and y is a multiple of 10, the term containing y^k is a multiple of 10^k. So the last j digits of $(x+y)^n$ depend only on the terms with $k < j$. For example,

$$21^5 = (1+20)^5 = 1 + 5\cdot 20 + (\text{terms containing } 20^2 \text{ or a higher power of } 20).$$

Every term in parentheses is a multiple of $20^2 = 400$, so it is a multiple of 100. Hence 21^5 ends in the same two digits as $1 + 100 = 101$, namely 01. Indeed, $21^5 = 4084101$.

Example.

Problem 4.2.1 (1971 AHSME, Problem #13)**.** If $(1.0025)^{10}$ is evaluated correct to 5 decimal places, then the digit in the fifth decimal place is:

(A) 0 **(B)** 1 **(C)** 2 **(D)** 5 **(E)** 8

Strategy. Expand $(1+0.0025)^{10}$. The terms shrink so quickly that the first four of them decide the first five decimal places.

Solution. By the binomial theorem,

$$(1.0025)^{10} = (1+0.0025)^{10} = 1 + 10(0.0025) + 45(0.0025)^2 + 120(0.0025)^3 + R,$$

where R is the sum of the other seven terms, $\binom{10}{k}(0.0025)^k$ for $k = 4, 5, \ldots, 10$. The first four terms add up to

$$1 + 0.025 + 0.00028125 + 0.000001875 = 1.025283125.$$

Each of the seven terms in R is positive and at most $\binom{10}{5}(0.0025)^4 = 252\cdot(0.0025)^4 < 10^{-8}$, because the largest entry of row 10 is its middle entry $\binom{10}{5} = 252$ and $(0.0025)^k \le (0.0025)^4$ for $k \ge 4$. So $0 < R < 7\cdot 10^{-8}$, and

$$1.025283125 < (1.0025)^{10} < 1.025283195.$$

Correct to 5 decimal places, $(1.0025)^{10} = 1.02528$, and the digit in the fifth decimal place is 8.

Answer: (E) 8. □

4.3. **Sums of Binomial Coefficients.**

Theorem 4.3.1 (Sum of a Row)**.** *For every nonnegative integer n,*

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n.$$

Proof. For $n \ge 1$, put $x = y = 1$ in the binomial theorem: the left side becomes 2^n, and each term on the right becomes its coefficient $\binom{n}{k}$. A second proof counts the subsets of an n-element set. Each element is either in a subset or not, so by the Multiplication Principle there are 2^n subsets. Grouping the subsets by their numbers of elements gives $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}$ subsets. □

Corollary 4.3.2. *An n-element set has 2^n subsets, including the empty set and the whole set.*

For example, a 5-element set has $1+5+10+10+5+1 = 32 = 2^5$ subsets. More generally, the sum of the numerical coefficients of an expansion is its value when every variable is replaced by 1, because each term then becomes its coefficient.

Example.

Problem 4.3.3 (1964 AHSME, Problem #20)**.** The sum of the numerical coefficients of all the terms in the expansion of $(x-2y)^{18}$ is:

(A) 0 **(B)** 1 **(C)** 19 **(D)** -1 **(E)** -19

Strategy. Replace both variables by 1.

Solution. Replacing x and y by 1 turns every term of the expansion into its numerical coefficient. So the sum of the coefficients is

$$(1-2\cdot 1)^{18} = (-1)^{18} = 1.$$

Answer: (B) 1. □

5. Lattice Paths

A lattice path moves along the lines of a grid. Counting such paths combines the ideas of the previous sections.

5.1. **Counting Paths.**

Definition 5.1.1. Let a and b be nonnegative integers. A **lattice path** from $(0,0)$ to (a,b) moves along the grid lines in unit steps, each one unit to the right (R) or one unit up (U). On a map of streets, the steps are blocks east and blocks north.

Theorem 5.1.2. *The number of lattice paths from* $(0,0)$ *to* (a,b) *is*

$$\binom{a+b}{a} = \binom{a+b}{b}.$$

Proof. Every such path consists of exactly a R-steps and b U-steps. Writing its steps in order gives a row of a R's and b U's, and each such row describes exactly one path. By Remark 1.1.3, there are $\binom{a+b}{a}$ such rows: choose which a of the $a+b$ steps are R. □

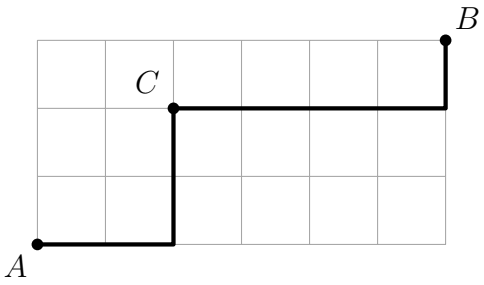
Only the differences of the coordinates matter: if $r \geq p$ and $s \geq q$, the number of lattice paths from (p,q) to (r,s) is

$$\binom{(r-p)+(s-q)}{r-p}.$$

A path through a point C is a path to C followed by a path from C, so by the Multiplication Principle the number of paths through C is the number of paths to C times the number of paths from C.

Example.

Problem 5.1.3. In the grid below, a path goes from A to B along the grid lines, one unit at a time, with each step to the right or up. One such path is shown.



(a) How many such paths are there?
(b) How many of them pass through C?

Strategy. Each path is a row of 6 R's and 3 U's. For (b), split the path at C.

Solution. (a) A path from A to B makes 6 steps to the right and 3 steps up, in some order, so there are $\binom{9}{3} = 84$ paths.

(b) From A to C a path makes 2 steps to the right and 2 steps up, so there are $\binom{4}{2} = 6$ paths from A to C. From C to B it makes 4 steps to the right and 1 step up, so there are $\binom{5}{1} = 5$ paths from C to B. By the Multiplication Principle, $6 \cdot 5 = 30$ paths pass through C. The path shown is RRUURRRRU, one of these 30.

Answer: (a) 84 and (b) 30. □

5.2. **Paths with Obstacles.** The last step of a path to (x, y) comes from $(x-1, y)$ or from $(x, y-1)$. So the number of paths to (x, y) is the number of paths to $(x-1, y)$ plus the number of paths to $(x, y-1)$, by the Addition Principle. To **count point by point**, write 1 at the starting point and fill in the other points in order, each with the sum of the numbers to its left and below it.

With no obstacles this rule is Pascal's identity,

$$\binom{x+y}{x} = \binom{x+y-1}{x-1} + \binom{x+y-1}{x},$$

and the numbers are the entries of Pascal's triangle turned on its side. A point that may not be used gets the number 0, and points outside the allowed region are left out. The same method works for any region and for other kinds of steps: the number at a point is the sum of the numbers at all the points from which its last step can come.

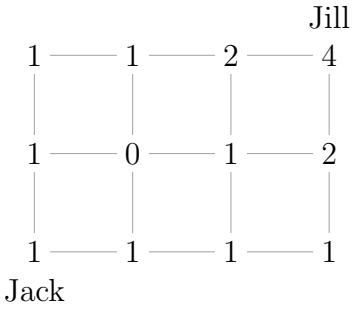
Example.

Problem 5.2.1 (2014 AMC 8, Problem #11)**.** Jack wants to bike from his house to Jill's house, which is located three blocks east and two blocks north of Jack's house. After biking each block, Jack can continue either east or north, but he needs to avoid a dangerous intersection one block east and one block north of his house. In how many ways can he reach Jill's house by biking a total of five blocks?

(A) 4 **(B)** 5 **(C)** 6 **(D)** 8 **(E)** 10

Strategy. Put Jack's house at $(0, 0)$ and count point by point, writing 0 at the dangerous intersection.

Solution. Put Jack's house at $(0, 0)$, with east to the right and north up. Then Jill's house is at $(3, 2)$ and the dangerous intersection is at $(1, 1)$. A trip of five blocks to Jill's house must use exactly 3 blocks east and 2 blocks north, so it is a lattice path from $(0, 0)$ to $(3, 2)$ that avoids $(1, 1)$. Counting point by point, with 0 at $(1, 1)$, gives the numbers below.



For example, the number at $(2, 1)$ is $0 + 1 = 1$, the sum of the numbers at $(1, 1)$ and $(2, 0)$. The number at Jill's house is 4.

Answer: (A) 4. □

6. Problems

Beginner.

1. (2017 AMC 10A, Problem #8) At a gathering of 30 people, there are 20 people who all know each other and 10 people who know no one. People who know each other hug, and people who do not know each other shake hands. How many handshakes occur?
(A) 240 **(B)** 245 **(C)** 290 **(D)** 480 **(E)** 490

2. (2004 AMC 10A, Problem #12) Henry's Hamburger Heaven offers its hamburgers with the following condiments: ketchup, mustard, mayonnaise, tomato, lettuce, pickles, cheese, and onions. A customer can choose one, two, or three meat patties, and any collection of condiments. How many different kinds of hamburgers can be ordered?
(A) 24 **(B)** 256 **(C)** 768 **(D)** 40,320 **(E)** 120,960

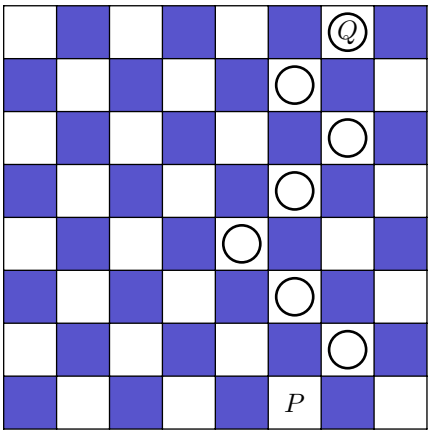
3. (2002 AMC 10B, Problem #18) Four distinct circles are drawn in a plane. What is the maximum number of points where at least two of the circles intersect?
(A) 8 **(B)** 9 **(C)** 10 **(D)** 12 **(E)** 16

4. (2013 AMC 8, Problem #21) Samantha lives 2 blocks west and 1 block south of the southwest corner of City Park. Her school is 2 blocks east and 2 blocks north of the northeast corner of City Park. On school days she bikes on streets to the southwest corner of City Park, then takes a diagonal path through the park to the northeast corner, and then bikes on streets to school. If her route is as short as possible, how many different routes can she take?
(A) 3 **(B)** 6 **(C)** 9 **(D)** 12 **(E)** 18

5. (1962 AHSME, Problem #12) When $(1 - \frac{1}{a})^6$ is expanded, the sum of the last three coefficients is:
(A) 22 **(B)** 11 **(C)** 10 **(D)** -10 **(E)** -11

Intermediate.

6. (2020 AMC 8, Problem #21) A game board consists of 64 squares that alternate between shaded and unshaded. The figure below shows square P in the bottom row and square Q in the top row. A marker is placed at P. A step consists of moving the marker onto one of the adjoining unshaded squares in the row above. How many 7-step paths are there from P to Q? (The figure shows a sample path.)



(A) 28 **(B)** 30 **(C)** 32 **(D)** 33 **(E)** 35

7. (1986 AHSME, Problem #23) Let

$$N = 69^5 + 5 \cdot 69^4 + 10 \cdot 69^3 + 10 \cdot 69^2 + 5 \cdot 69 + 1.$$

How many positive integers are factors of N?

(A) 3 **(B)** 5 **(C)** 69 **(D)** 125 **(E)** 216

8. (2007 AMC 10B, Problem #20) A set of 25 square blocks is arranged into a 5×5 square. How many different combinations of 3 blocks can be selected from that set so that no two are in the same row or column?

(A) 100 **(B)** 125 **(C)** 600 **(D)** 2300 **(E)** 3600

9. (2023 AMC 10B, Problem #16) Define an *upno* to be a positive integer of 2 or more digits where the digits are strictly increasing moving left to right. Similarly, define a *downno* to be a positive integer of 2 or more digits where the digits are strictly decreasing moving left to right. For instance, the number 258 is an upno and 8620 is a downno. Let U equal the total number of upnos and let D equal the total number of downnos. What is $|U - D|$?

(A) 512 **(B)** 10 **(C)** 0 **(D)** 9 **(E)** 511

10. (1997 AHSME, Problem #24) A *rising number*, such as 34689, is a positive integer each digit of which is larger than each of the digits to its left. There are $\binom{9}{5} = 126$ five-digit rising numbers. When these numbers are arranged from smallest to largest, the 97th number in the list does not contain the digit

(A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

Advanced.

11. (2012 AIME II, Problem #3) At a certain university, the division of mathematical sciences consists of the departments of mathematics, statistics, and computer science. There are two male and two female professors in each department. A committee of six professors is to contain three men and three women and must also contain two professors from each of the three departments. Find the number of possible committees that can be formed subject to these requirements.

12. (2010 AMC 10A, Problem #22) Eight points are chosen on a circle, and chords are drawn connecting every pair of points. No three chords intersect in a single point inside the circle. How many triangles with all three vertices in the interior of the circle are created?
(A) 28 **(B)** 56 **(C)** 70 **(D)** 84 **(E)** 140

13. (2024 AIME II, Problem #6) Alice chooses a set A of positive integers. Then Bob lists all finite nonempty sets B of positive integers with the property that the maximum element of B belongs to A. Bob's list has 2024 sets. Find the sum of the elements of A.

14. (2011 AMC 10B, Problem #23) What is the hundreds digit of 2011^{2011}?
(A) 1 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 9

15. (2022 AMC 10A, Problem #22) Suppose that 13 cards numbered 1, 2, 3, $\cdots$, 13 are arranged in a row. The task is to pick them up in numerically increasing order, working repeatedly from left to right. In the example below, cards 1, 2, 3 are picked up on the first pass, 4 and 5 on the second pass, 6 on the third pass, 7, 8, 9, 10 on the fourth pass, and 11, 12, 13 on the fifth pass. For how many of the 13! possible orderings of the cards will the 13 cards be picked up in exactly two passes?



(A) 4082 **(B)** 4095 **(C)** 4096 **(D)** 8178 **(E)** 8191