COMBINATIONS AND THE BINOMIAL THEOREM FOR AMC 10: SOLUTIONS

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1. Solutions

Beginner.

1. (2017 AMC 10A, Problem #8) At a gathering of 30 people, there are 20 people who all know each other and 10 people who know no one. People who know each other hug, and people who do not know each other shake hands. How many handshakes occur?

(A) 240 **(B)** 245 **(C)** 290 **(D)** 480 **(E)** 490

Strategy. Two people shake hands unless both of them are among the 20 people who know each other. Count the pairs that shake hands according to how many of the two people are among the other 10.

Solution. Call the 20 people who all know each other the first group and the 10 people who know no one the second group. Two people shake hands exactly when they do not know each other, that is, unless both of them are in the first group. Two people from the second group can be chosen in $\binom{10}{2} = 45$ ways. One person from each group can be chosen in $10 \cdot 20 = 200$ ways, by the Multiplication Principle. These two cases do not overlap, so there are $45 + 200 = 245$ handshakes.

Answer: (B) 245. □

2. (2004 AMC 10A, Problem #12) Henry's Hamburger Heaven offers its hamburgers with the following condiments: ketchup, mustard, mayonnaise, tomato, lettuce, pickles, cheese, and onions. A customer can choose one, two, or three meat patties, and any collection of condiments. How many different kinds of hamburgers can be ordered?

(A) 24 **(B)** 256 **(C)** 768 **(D)** 40,320 **(E)** 120,960

Strategy. A collection of condiments is a subset of the set of 8 condiments.

Solution. The number of meat patties can be chosen in 3 ways. A collection of condiments is a subset of the 8 condiments, and every subset is allowed, including the empty set (no condiments) and the whole set. An 8-element set has $2^8 = 256$ subsets, so there are 256 collections of condiments. By the Multiplication Principle there are $3 \cdot 256 = 768$ kinds of hamburgers.

Answer: (C) 768. □

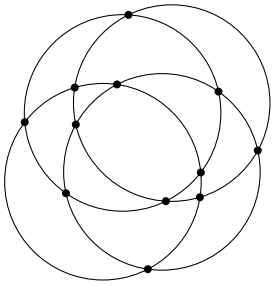
3. (2002 AMC 10B, Problem #18) Four distinct circles are drawn in a plane. What is the maximum number of points where at least two of the circles intersect?

(A) 8 **(B)** 9 **(C)** 10 **(D)** 12 **(E)** 16

Strategy. Every point where two of the circles intersect lies on some pair of circles. Bound the number of points each pair can give, and then find four circles that reach the bound.

Solution. Three points lie on at most one circle, so two different circles share at most 2 points. Every point where at least two of the four circles intersect is a common point of one of the pairs of circles, and the number of pairs is $\binom{4}{2} = 6$. So there are no more than $6 \cdot 2 = 12$ such points.

Twelve points are possible. Take the four circles of radius 1 centered at $(0,0)$, $(0.6,0.1)$, $(0.2,0.7)$ and $(0.7,0.8)$, shown below. Any two of the centers are less than 2 apart, so every two of the circles meet in 2 points. A point coming from two different pairs of circles would lie on at least three of the circles, and the figure shows that no point lies on three of them. So the 12 points are all different.



Answer: (D) 12. □

4. (2013 AMC 8, Problem #21) Samantha lives 2 blocks west and 1 block south of the southwest corner of City Park. Her school is 2 blocks east and 2 blocks north of the northeast corner of City Park. On school days she bikes on streets to the southwest corner of City Park, then takes a diagonal path through the park to the northeast corner, and then bikes on streets to school. If her route is as short as possible, how many different routes can she take?

(A) 3 **(B)** 6 **(C)** 9 **(D)** 12 **(E)** 18

Strategy. The route has three parts. Count the shortest routes for each part, and multiply.

Solution. A shortest route from Samantha's home to the southwest corner of the park goes 2 blocks east and 1 block north in some order, so it is a lattice path of 3 blocks, 1 of them north. There are $\binom{3}{1} = 3$ such routes. The diagonal path through the park is a single path. A shortest route from the northeast corner of the park to school goes 2 blocks east and 2 blocks north in some order, so there are $\binom{4}{2} = 6$ such routes. By the Multiplication Principle there are $3 \cdot 1 \cdot 6 = 18$ routes.

Answer: (E) 18. □

5. (1962 AHSME, Problem #12) When $(1 - \frac{1}{a})^6$ is expanded, the sum of the last three coefficients is:

(A) 22 **(B)** 11 **(C)** 10 **(D)** -10 **(E)** -11

Strategy. Write out the expansion with the binomial theorem, keeping track of the signs.

Solution. By the binomial theorem with 1 in place of x and $-1/a$ in place of y, the term containing $(-1/a)^k$ is $\binom{6}{k}(-1/a)^k$, so

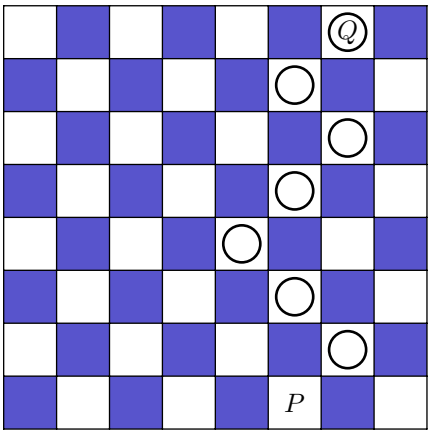
$$\left(1 - \frac{1}{a}\right)^6 = 1 - \frac{6}{a} + \frac{15}{a^2} - \frac{20}{a^3} + \frac{15}{a^4} - \frac{6}{a^5} + \frac{1}{a^6}.$$

The coefficients are $1, -6, 15, -20, 15, -6, 1$. This list reads the same in both directions, so the last three coefficients are 15, -6 and 1 whichever order the terms are written in. Their sum is $15 - 6 + 1 = 10$.

Answer: (C) 10. □

Intermediate.

6. (2020 AMC 8, Problem #21) A game board consists of 64 squares that alternate between shaded and unshaded. The figure below shows square P in the bottom row and square Q in the top row. A marker is placed at P. A step consists of moving the marker onto one of the adjoining unshaded squares in the row above. How many 7-step paths are there from P to Q? (The figure shows a sample path.)

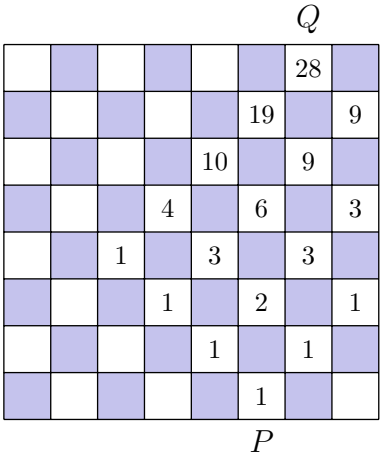


(A) 28 **(B)** 30 **(C)** 32 **(D)** 33 **(E)** 35

Strategy. Find the squares the marker can move to in one step. Then count point by point, writing in each square the number of ways to reach it from P.

Solution. Shaded and unshaded squares alternate, so the square directly above an unshaded square is shaded, and the unshaded squares that adjoin it in the row above are the squares diagonally above it. So every step goes diagonally to the next row, and the marker cannot leave the board.

Write in each square the number of paths from P to that square, starting with 1 in P. The last step into a square comes from one of the squares diagonally below it, so by the Addition Principle the number in a square is the sum of the numbers in the squares diagonally below it. This is counting point by point, with the edge of the board as an obstacle: a square at the left or right edge of the board has only one square diagonally below it. The figure shows the numbers in the squares that lie on some path from P to Q.



The number in Q is $19 + 9 = 28$.

Answer: (A) 28. □

7. (1986 AHSME, Problem #23) Let

$$N = 69^5 + 5 \cdot 69^4 + 10 \cdot 69^3 + 10 \cdot 69^2 + 5 \cdot 69 + 1.$$

How many positive integers are factors of N?

(A) 3 **(B)** 5 **(C)** 69 **(D)** 125 **(E)** 216

Strategy. The coefficients $1, 5, 10, 10, 5, 1$ are row 5 of Pascal's triangle.

Solution. The coefficients $1, 5, 10, 10, 5, 1$ are $\binom{5}{0}, \binom{5}{1}, \ldots, \binom{5}{5}$. By the binomial theorem with $x = 69$ and $y = 1$,

$$N = \binom{5}{0}69^5 + \binom{5}{1}69^4 + \binom{5}{2}69^3 + \binom{5}{3}69^2 + \binom{5}{4}69 + \binom{5}{5} = (69+1)^5 = 70^5.$$

Since $70 = 2 \cdot 5 \cdot 7$, $N = 2^5 \cdot 5^5 \cdot 7^5$. A positive factor of N has the form $2^a \cdot 5^b \cdot 7^c$, where each of a, b, c is one of $0, 1, \ldots, 5$, and different choices of the exponents give different factors. By the Multiplication Principle, N has $6 \cdot 6 \cdot 6 = 216$ positive factors.

Answer: (E) 216. □

8. (2007 AMC 10B, Problem #20) A set of 25 square blocks is arranged into a 5×5 square. How many different combinations of 3 blocks can be selected from that set so that no two are in the same row or column?

(A) 100 **(B)** 125 **(C)** 600 **(D)** 2300 **(E)** 3600

Strategy. Choose the three rows and the three columns that the blocks occupy, and then match the rows with the columns.

Solution. The three blocks lie in three different rows, which can be chosen in $\binom{5}{3} = 10$ ways, and in three different columns, which can also be chosen in 10 ways. Each of the chosen rows contains exactly one of the blocks, and so does each of the chosen columns. So once the rows and the columns are chosen, the blocks are determined by matching the three rows with the three columns. List the chosen rows from top to bottom: the first can be matched with any of the 3 chosen columns, the second with either of the other 2, and the third with the remaining column, in $3! = 6$ ways in all. By the Multiplication Principle there are $10 \cdot 10 \cdot 6 = 600$ sets of blocks.

As a check, choosing the blocks one at a time gives $25 \cdot 16 \cdot 9 = 3600$ ordered choices, since the second block must avoid the row and the column of the first, and the third must avoid the rows and the columns of the first two. Each set of 3 blocks is counted $3! = 6$ times, once for each order, and $3600/6 = 600$.

Answer: (C) 600. □

9. (2023 AMC 10B, Problem #16) Define an *upno* to be a positive integer of 2 or more digits where the digits are strictly increasing moving left to right. Similarly, define a *downno* to be a positive integer of 2 or more digits where the digits are strictly decreasing moving left to right. For instance, the number 258 is an upno and 8620 is a downno. Let U equal the total number of upnos and let D equal the total number of downnos. What is $|U - D|$?

(A) 512 **(B)** 10 **(C)** 0 **(D)** 9 **(E)** 511

Strategy. Each upno and each downno is determined by its set of digits. Compare the sets of digits that the two kinds of numbers can have.

Solution. An upno is determined by its set of digits: writing the digits of the set in increasing order gives the upno back. The digit 0 cannot appear in an upno, since it would

have to be the first digit. So the upnos correspond to the subsets of $\{1, 2, \ldots, 9\}$ with at least 2 elements. In the same way, a downno is determined by its set of digits, written in decreasing order, and here 0 may appear, as the last digit. So the downnos correspond to the subsets of $\{0, 1, \ldots, 9\}$ with at least 2 elements.

The downnos without the digit 0 have exactly the same sets of digits as the upnos, so there are U of them. A downno with the digit 0 has a set of digits $\{0\} \cup S$, where S is a nonempty subset of $\{1, 2, \ldots, 9\}$, and each such S gives one downno, even when S has only one element (for example, $S = \{7\}$ gives 70). A 9-element set has $2^9 = 512$ subsets, so there are $512 - 1 = 511$ choices of S. Hence $D = U + 511$, and $|U - D| = 511$. In fact, $U = 2^9 - 1 - 9 = 502$ and $D = 2^{10} - 1 - 10 = 1013$.

Answer: (E) 511. □

10. (1997 AHSME, Problem #24) A *rising number*, such as 34689, is a positive integer each digit of which is larger than each of the digits to its left. There are $\binom{9}{5} = 126$ five-digit rising numbers. When these numbers are arranged from smallest to largest, the 97th number in the list does not contain the digit

(A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

Strategy. Group the rising numbers in blocks by their first digit, then by their first two digits, and so on, and find the block that contains the 97th number.

Solution. A five-digit rising number is a 5-element subset of $\{1, 2, \ldots, 9\}$ written in increasing order. In the list from smallest to largest, the numbers are ordered by their first digit, then by their second digit, and so on.

The numbers beginning with 1 have their other four digits chosen from $\{2, 3, \ldots, 9\}$, so there are $\binom{8}{4} = 70$ of them, in positions 1 to 70. The numbers beginning with 2 have their other four digits chosen from $\{3, 4, \ldots, 9\}$, so there are $\binom{7}{4} = 35$ of them, in positions 71 to 105. So the 97th number begins with 2.

Among the numbers beginning with 2, the $\binom{6}{3} = 20$ numbers beginning with 23 come first, in positions 71 to 90. Next come the $\binom{5}{3} = 10$ numbers beginning with 24, in positions 91 to 100, so the 97th number begins with 24. Among these, the $\binom{4}{2} = 6$ numbers beginning with 245 are in positions 91 to 96. So the 97th number is the smallest one beginning with 246, which is 24678. It contains the digits 4, 6, 7 and 8, but not 5.

Answer: (B) 5. □

Advanced.

11. (2012 AIME II, Problem #3) At a certain university, the division of mathematical sciences consists of the departments of mathematics, statistics, and computer science. There are two male and two female professors in each department. A committee of six professors is to contain three men and three women and must also contain two professors from each of the three departments. Find the number of possible committees that can be formed subject to these requirements.

Strategy. Each department sends two professors. Use cases on the numbers of men sent by the three departments.

Solution. A department can send 2 men, in $\binom{2}{2} = 1$ way, 1 man and 1 woman, in $\binom{2}{1}\binom{2}{1} = 4$ ways, or 2 women, in 1 way. The committee has 6 professors, so it has 3 men and 3 women exactly when the numbers of men sent by the three departments add up to 3. Each of these numbers is 0, 1 or 2, so they are 1, 1, 1 or, in some order, 2, 1, 0.

Case 1: each department sends 1 man and 1 woman. There are $4 \cdot 4 \cdot 4 = 64$ committees.

Case 2: the numbers of men are 2, 1 and 0 in some order. There are $3! = 6$ ways to decide which department sends 2 men, which sends 1 man and which sends none, and then $1 \cdot 4 \cdot 1 = 4$ ways to choose the professors, so there are $6 \cdot 4 = 24$ committees.

By the Addition Principle there are $64 + 24 = 88$ committees.

Answer: 88. □

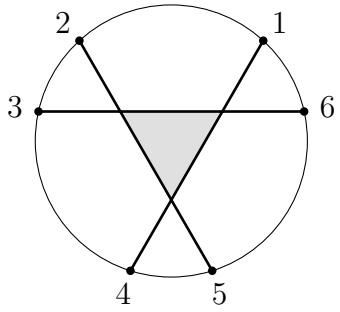
12. (2010 AMC 10A, Problem #22) Eight points are chosen on a circle, and chords are drawn connecting every pair of points. No three chords intersect in a single point inside the circle. How many triangles with all three vertices in the interior of the circle are created?

(A) 28 **(B)** 56 **(C)** 70 **(D)** 84 **(E)** 140

Strategy. Show that each such triangle is determined by 6 of the 8 points.

Solution. The sides of such a triangle lie on three of the chords, and each vertex is a point where two of these chords cross inside the circle. So the three chords cross one another in pairs inside the circle. Two chords with a common endpoint meet only at that endpoint, which is on the circle, so the three chords have 6 different endpoints.

Conversely, take any 6 of the 8 points and number them 1 to 6 in order around the circle. Consider three chords whose endpoints are these six points. They cross one another in pairs exactly when each of them has one endpoint of each of the other two chords on each side of it. In particular, each of them then has exactly 2 of the other 4 points on each side. The only such chords are the chords joining 1 to 4, 2 to 5 and 3 to 6, shown below. These three chords do cross in pairs, at three different points since no three chords meet in one point, so they form exactly one triangle with all three vertices inside the circle.



So the triangles correspond to the 6-element subsets of the 8 points, and the number of triangles is

$$\binom{8}{6} = \binom{8}{2} = 28.$$

Answer: (A) 28. □

13. (2024 AIME II, Problem #6) Alice chooses a set A of positive integers. Then Bob lists all finite nonempty sets B of positive integers with the property that the maximum element of B belongs to A. Bob's list has 2024 sets. Find the sum of the elements of A.

Strategy. Count Bob's sets according to their maximum element. Each count is a power of 2, so 2024 must be written as a sum of different powers of 2.

Solution. Removing the maximum element m from a finite nonempty set B of positive integers leaves a subset of $\{1, 2, \ldots, m-1\}$, possibly empty. Conversely, adding m to any subset of $\{1, 2, \ldots, m-1\}$ gives a finite nonempty set with maximum element m, and

different subsets give different sets. An $(m-1)$-element set has 2^{m-1} subsets, so exactly 2^{m-1} sets have maximum element m. Each set that Bob lists has exactly one maximum element, and it belongs to A, so by the Addition Principle the number of sets on the list is the sum of 2^{m-1} over all m in A. Since the list is finite, A is finite, and the numbers 2^{m-1} for m in A are different powers of 2 that add up to 2024.

A positive integer can be written as a sum of different powers of 2 in only one way, given by its base-2 representation. The largest power used for 2024 must be $2^{10} = 1024$: the next power, $2^{11} = 2048$, is too large, and all the smaller powers together add up to only $1 + 2 + \cdots + 2^9 = 2^{10} - 1 = 1023$. The same reasoning for $2024 - 1024 = 1000$, and so on, gives

$$2024 = 2^{10} + 2^9 + 2^8 + 2^7 + 2^6 + 2^5 + 2^3,$$

that is, $2024 = 11111101000_2$. So the numbers $m-1$ for m in A are 10, 9, 8, 7, 6, 5 and 3, and $A = \{4, 6, 7, 8, 9, 10, 11\}$. The sum of the elements of A is $4+6+7+8+9+10+11 = 55$.

Answer: 55. □

14. (2011 AMC 10B, Problem #23) What is the hundreds digit of 2011^{2011}?

(A) 1 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 9

Strategy. The hundreds digit depends only on the last three digits. Replace 2011 by 11, and then expand $(1+10)^{2011}$.

Solution. Write $2011^{2011} = (11 + 2000)^{2011}$. By the binomial theorem, every term of the expansion except 11^{2011} contains a factor 2000, which is a multiple of 1000. So 2011^{2011} and 11^{2011} have the same last three digits. Next, by the binomial theorem again,

$$11^{2011} = (1+10)^{2011} = 1 + 2011 \cdot 10 + \binom{2011}{2} \cdot 10^2 + R,$$

where every term of R contains 10^3 or a higher power of 10, so R is a multiple of 1000. The number $2011 \cdot 10 = 20110$ ends in 110. Also

$$\binom{2011}{2} = \frac{2011 \cdot 2010}{2} = 2011 \cdot 1005 = 2021055,$$

so $\binom{2011}{2} \cdot 10^2 = 202105500$ ends in 500. Hence the last three digits of 11^{2011} are those of $1 + 110 + 500 = 611$, and the hundreds digit of 2011^{2011} is 6.

Answer: (D) 6. □

15. (2022 AMC 10A, Problem #22) Suppose that 13 cards numbered 1, 2, 3, $\cdots$, 13 are arranged in a row. The task is to pick them up in numerically increasing order, working repeatedly from left to right. In the example below, cards 1, 2, 3 are picked up on the first pass, 4 and 5 on the second pass, 6 on the third pass, 7, 8, 9, 10 on the fourth pass, and 11, 12, 13 on the fifth pass. For how many of the 13! possible orderings of the cards will the 13 cards be picked up in exactly two passes?

7	11	8	6	4	5	9	12	1	13	10	2	3

(A) 4082 **(B)** 4095 **(C)** 4096 **(D)** 8178 **(E)** 8191

Strategy. If the first pass picks up $1, 2, \ldots, k$, the cards $1, \ldots, k$ and the cards $k+1, \ldots, 13$ each appear in increasing order. Count such orderings with a binomial coefficient, and add over k.

Solution. Suppose the cards are picked up in exactly two passes, and the first pass picks up the cards $1, 2, \ldots, k$, where $1 \le k \le 12$. Then the second pass picks up $k+1, \ldots, 13$. The first pass picks up its cards from left to right, so the cards $1, \ldots, k$ appear in increasing order from left to right, and in the same way the cards $k+1, \ldots, 13$ appear in increasing order. Also, card $k+1$ is to the left of card k, since otherwise the first pass would continue with card $k+1$.

Conversely, suppose that for some k with $1 \le k \le 12$, the cards $1, \ldots, k$ appear in increasing order, the cards $k+1, \ldots, 13$ appear in increasing order, and card $k+1$ is to the left of card k. Then the first pass picks up exactly the cards $1, \ldots, k$, and the second pass picks up all the others, so the cards are picked up in exactly two passes. Here k is the number of cards picked up on the first pass, so each such ordering arises from only one value of k.

Fix k. An ordering in which both groups of cards appear in increasing order is determined by the set of k positions holding the cards $1, \ldots, k$: fill these positions with $1, \ldots, k$ from left to right and the other positions with $k+1, \ldots, 13$ from left to right. So there are $\binom{13}{k}$ such orderings. In exactly one of them, card k is to the left of card $k+1$: since k is the last card of its group and $k+1$ is the first card of the other group, this happens only when all of $1, \ldots, k$ come before all of $k+1, \ldots, 13$, which is the ordering $1, 2, \ldots, 13$. So exactly $\binom{13}{k} - 1$ orderings are picked up in two passes with k cards on the first pass. By the Addition Principle and the Sum of a Row, the number of orderings is

$$\sum_{k=1}^{12} \left(\binom{13}{k} - 1 \right) = \left(2^{13} - \binom{13}{0} - \binom{13}{13} \right) - 12 = 8190 - 12 = 8178.$$

Answer: (D) 8178. □

2. Answer Key

Problem	Letter	Value
1	B	245
2	C	768
3	D	12
4	E	18
5	C	10
6	A	28
7	E	216
8	C	600
9	E	511
10	B	5
11		88
12	A	28
13		55
14	D	6
15	D	8178