COMBINATIONS AND THE BINOMIAL THEOREM FOR AMC 10: PRACTICE SET SOLUTIONS

ANDY JO

Contents

1. Solutions 2
2. Answer Key 11

1. Solutions

Beginner.

1. (2013 AMC 10A, Problem #7) A student must choose a program of four courses from a menu of courses consisting of English, Algebra, Geometry, History, Art, and Latin. This program must contain English and at least one mathematics course. In how many ways can this program be chosen?

(A) 6 **(B)** 8 **(C)** 9 **(D)** 12 **(E)** 16

Strategy. English is in every program, so only the other three courses need to be chosen. Count all such choices, then subtract the ones with no mathematics course.

Solution. Every program contains English, so a program is determined by choosing its other 3 courses from the remaining 5: Algebra, Geometry, History, Art, and Latin. This can be done in $\binom{5}{3} = 10$ ways. The program must also contain at least one mathematics course, Algebra or Geometry. The only choice of 3 courses with neither is History, Art, and Latin. Therefore the number of programs is $10 - 1 = 9$.

Answer: (C) 9. □

2. (1950 AHSME, Problem #45) The number of diagonals that can be drawn in a polygon of 100 sides is:

(A) 4850 **(B)** 4950 **(C)** 9900 **(D)** 98 **(E)** 8800

Strategy. A diagonal joins two vertices that are not neighbors, and it is determined by its two endpoints. Count the pairs of vertices, then remove the pairs that form sides.

Solution. A segment joining two of the 100 vertices is determined by its set of two endpoints, so there are

$$\binom{100}{2} = \frac{100 \cdot 99}{2} = 4950$$

such segments. Such a segment is a side of the polygon when its endpoints are neighboring vertices, and a diagonal otherwise. The polygon has 100 sides, so it has $4950 - 100 = 4850$ diagonals.

Ordered choices give the same count. Each vertex is joined by a diagonal to every vertex other than itself and its two neighbors, that is, to 97 vertices. The product $100 \cdot 97 = 9700$ counts each diagonal twice, once from each endpoint, so there are $9700/2 = 4850$ diagonals.

Answer: (A) 4850. □

3. (2007 AMC 10B, Problem #8) On the trip home from the meeting where this AMC 10 was constructed, the Contest Chair noted that his airport parking receipt had digits of the form *bbcac*, where $0 \le a < b < c \le 9$, and b was the average of a and c. How many different five-digit numbers satisfy all these properties?

(A) 12 **(B)** 16 **(C)** 18 **(D)** 20 **(E)** 24

Strategy. The number is determined by the set $\{a, c\}$. The average of two different digits always lies strictly between them, so the only condition is that it be a whole number, and that depends on whether a and c are even or odd.

Solution. Since b is the average of a and c, the number *bbcac* is determined by the two digits $a < c$, that is, by the set $\{a, c\}$, and different sets give different numbers. Conversely, take two digits $a < c$. Their average $\frac{a+c}{2}$ lies strictly between a and c, so it is a digit b with $a < b < c$ exactly when it is a whole number, that is, when $a + c$ is even. Then $b > a \ge 0$,

so the first digit b is not 0, and $bbcac$ is a five-digit number. The sum $a+c$ is even exactly when a and c are both even or both odd, which gives two cases.

Case 1: a and c are even. The set $\{a, c\}$ is a 2-element subset of $\{0, 2, 4, 6, 8\}$, so there are $\binom{5}{2} = 10$ numbers.

Case 2: a and c are odd. The set $\{a, c\}$ is a 2-element subset of $\{1, 3, 5, 7, 9\}$, so there are again $\binom{5}{2} = 10$ numbers.

The cases do not overlap, so by the Addition Principle there are $10 + 10 = 20$ numbers.

Answer: (D) 20. □

4. (1961 AHSME, Problem #7) When simplified, the third term in the expansion of

$$\left(\frac{a}{\sqrt{x}} - \frac{\sqrt{x}}{a^2}\right)^6$$

is:

(A) $\frac{15}{x}$ **(B)** $-\frac{15}{x}$ **(C)** $-\frac{6x^2}{a^9}$ **(D)** $\frac{20}{a^3}$ **(E)** $-\frac{20}{a^3}$

Strategy. Use the binomial theorem with the two fractions as the two terms of the binomial. The third term contains the square of the second fraction.

Solution. Write $u = a/\sqrt{x}$ and $v = -\sqrt{x}/a^2$. By the binomial theorem,

$$(u+v)^6 = u^6 + \binom{6}{1}u^5v + \binom{6}{2}u^4v^2 + \cdots + v^6,$$

so the third term is $\binom{6}{2}u^4v^2 = 15u^4v^2$. Since $(\sqrt{x})^4 = x^2$ and $(\sqrt{x})^2 = x$,

$$u^4 = \frac{a^4}{x^2} \qquad \text{and} \qquad v^2 = \frac{x}{a^4}.$$

Hence the third term is

$$15 \cdot \frac{a^4}{x^2} \cdot \frac{x}{a^4} = \frac{15}{x}.$$

Answer: (A) $\frac{15}{x}$. □

5. (2005 AMC 10B, Problem #18) All of David's telephone numbers have the form 555–abc–$defg$, where a, b, c, d, e, f, and g are distinct digits and in increasing order, and none is either 0 or 1. How many different telephone numbers can David have?

(A) 1 **(B)** 2 **(C)** 7 **(D)** 8 **(E)** 9

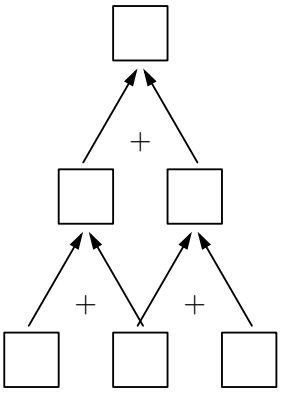
Strategy. The digits must appear in increasing order, so once the set of seven digits is chosen, the telephone number is determined.

Solution. The seven digits a, b, c, d, e, f, g are distinct and none is 0 or 1, so they are seven of the eight digits $2, 3, \ldots, 9$. Because they appear in increasing order, each choice of seven digits gives exactly one telephone number, and different choices give different numbers. So the number of telephone numbers is $\binom{8}{7} = 8$. Equivalently, a telephone number is determined by the one digit from 2 to 9 that it leaves out.

Answer: (D) 8. □

Intermediate.

6. (2006 AMC 8, Problem #22) Three different one-digit positive integers are placed in the bottom row of cells. Numbers in adjacent cells are added and the sum is placed in the cell above them. In the second row, continue the same process to obtain a number in the top cell. What is the difference between the largest and smallest numbers possible in the top cell?



(A) 16 **(B)** 24 **(C)** 25 **(D)** 26 **(E)** 35

Strategy. Name the bottom numbers and write the top number in terms of them. The cells are filled by the rule that builds Pascal's triangle, so the middle number counts twice.

Solution. Let a, b and c be the numbers in the bottom row, from left to right. The second row holds $a+b$ and $b+c$, so the top cell holds

$$(a+b)+(b+c)=a+2b+c.$$

The coefficients 1, 2, 1 form row 2 of Pascal's triangle, because each cell is the sum of the two cells below it, just as, by Pascal's identity, each entry inside a row of the triangle is the sum of the two entries above it.

Write the top number as $b+(a+b+c)$. The numbers a, b and c are three different integers from 1 to 9, so $b\le 9$ and $a+b+c\le 9+8+7=24$. Hence the top number is at most $9+24=33$, and it equals 33 when $b=9$, $a=8$ and $c=7$. In the same way, $b\ge 1$ and $a+b+c\ge 1+2+3=6$, so the top number is at least $1+6=7$, and it equals 7 when $b=1$, $a=2$ and $c=3$. The difference between the largest and smallest numbers is $33-7=26$.

Answer: (D) 26. □

7. (1975 AHSME, Problem #5) The polynomial $(x+y)^9$ is expanded in decreasing powers of x. The second and third terms have equal values when evaluated at $x=p$ and $y=q$, where p and q are positive numbers whose sum is one. What is the value of p?

(A) $\frac{1}{5}$ **(B)** $\frac{4}{5}$ **(C)** $\frac{1}{4}$ **(D)** $\frac{3}{4}$ **(E)** $\frac{8}{9}$

Strategy. In decreasing powers of x, the second and third terms are the terms containing y and y^2. Set their values equal and use $p+q=1$.

Solution. By the binomial theorem, written in decreasing powers of x, the $(k+1)$st term of the expansion of $(x+y)^9$ is the term containing y^k, namely $\binom{9}{k}x^{9-k}y^k$. So the second and third terms are

$$\binom{9}{1}x^8y=9x^8y \qquad \text{and} \qquad \binom{9}{2}x^7y^2=36x^7y^2.$$

At $x = p$ and $y = q$ they have equal values when

$$9p^8q = 36p^7q^2.$$

Since p and q are positive, dividing both sides by $9p^7q$ gives $p = 4q$. Then $p + q = 1$ becomes $5q = 1$, so $q = \frac{1}{5}$ and $p = \frac{4}{5}$.

Answer: (B) $\frac{4}{5}$. □

8. (1994 AHSME, Problem #15) For how many n in $\{1, 2, 3, \ldots, 100\}$ is the tens digit of n^2 odd?

(A) 10 **(B)** 20 **(C)** 30 **(D)** 40 **(E)** 50

Strategy. Write n as its units digit plus a multiple of 10, and expand the square with the binomial theorem. The last two digits of n^2 depend only on the first two terms, and the second term adds an even number of tens.

Solution. Write $n = b + 10a$, where b is the units digit of n and a is a nonnegative integer. By the binomial theorem,

$$n^2 = (b + 10a)^2 = b^2 + 2 \cdot b \cdot 10a + (10a)^2 = b^2 + 20ab + 100a^2.$$

The term $100a^2$ is a multiple of 100, so the last two digits of n^2 are those of $b^2 + 20ab$. Let t and u be the tens and units digits of b^2, which is at most 81, so that $b^2 = 10t + u$. Then

$$b^2 + 20ab = u + 10(t + 2ab),$$

so the tens digit of n^2 is the units digit of $t + 2ab$. Since $2ab$ is even, this digit is odd exactly when t is odd.

The squares of the digits are $0, 1, 4, 9, 16, 25, 36, 49, 64, 81$, and only 16 and 36 have an odd tens digit. So the tens digit of n^2 is odd exactly when n ends in 4 or 6. Among $1, 2, \ldots, 100$, ten numbers end in 4 and ten end in 6, so there are 20 such n.

Answer: (B) 20. □

9. (1989 AIME, Problem #2) Ten points are marked on a circle. How many distinct convex polygons of three or more sides can be drawn using some (or all) of the ten points as vertices?

Strategy. A convex polygon with its vertices among the points is determined by its set of vertices, which can be any set of at least 3 of the points. Count these sets with the sum of a row of Pascal's triangle.

Solution. Each polygon drawn has a set of at least 3 of the ten points as its vertices. Conversely, take any set of at least 3 of the points. Joining them in the order in which they lie around the circle gives a convex polygon with exactly these vertices, and it is the only one: joining the same points in any other order makes two sides cross. So the convex polygons correspond to the subsets of the ten points with at least 3 elements, and their number is

$$\binom{10}{3} + \binom{10}{4} + \cdots + \binom{10}{10}.$$

By the Sum of a Row, all of row 10 of Pascal's triangle adds up to $2^{10} = 1024$, so this sum is

$$2^{10} - \binom{10}{0} - \binom{10}{1} - \binom{10}{2} = 1024 - 1 - 10 - 45 = 968.$$

Answer: 968. □

10. (2005 AIME II, Problem #1) A game uses a deck of n different cards, where n is an integer and $n \geq 6$. The number of possible sets of 6 cards that can be drawn from the deck is 6 times the number of possible sets of 3 cards that can be drawn. Find n.

Strategy. The condition says $\binom{n}{6} = 6\binom{n}{3}$. Write both sides as products of decreasing factors and cancel.

Solution. A set of cards drawn from the deck is a subset of the n cards, so the condition is $\binom{n}{6} = 6\binom{n}{3}$. Writing each binomial coefficient as a product of decreasing factors divided by a factorial gives

$$\frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{720} = 6 \cdot \frac{n(n-1)(n-2)}{6} = n(n-1)(n-2).$$

Since $n \geq 6$, the product $n(n-1)(n-2)$ is not 0. Dividing both sides by it and multiplying by 720 gives

$$(n-3)(n-4)(n-5) = 720 = 10 \cdot 9 \cdot 8.$$

The left side is a product of three consecutive positive integers, and it increases with n, so it equals $10 \cdot 9 \cdot 8$ only when $n - 3 = 10$. Hence $n = 13$. As a check, a deck of 13 cards has 1716 sets of 6 cards and 286 sets of 3 cards, and $1716 = 6 \cdot 286$.

Answer: 13. □

Advanced.

11. (2003 AMC 12A, Problem #20) How many 15-letter arrangements of 5 A's, 5 B's, and 5 C's have no A's in the first 5 letters, no B's in the next 5 letters, and no C's in the last 5 letters?

(A) $\displaystyle\sum_{k=0}^{5} \binom{5}{k}^3$ **(B)** $3^5 \cdot 2^5$ **(C)** 2^{15} **(D)** $\dfrac{15!}{(5!)^3}$ **(E)** 3^{15}

Strategy. Split the arrangement into three blocks of five letters. The number of B's in the first block fixes how many of each letter every block holds, each block is then a row of two kinds of letters, and symmetry makes the three counts equal.

Solution. Call the first 5 letters the first block, the next 5 the middle block and the last 5 the last block. The first block has no A's, so it consists of some number k of B's and $5 - k$ C's, where $0 \leq k \leq 5$. The middle block has no B's, so the other $5 - k$ B's are all in the last block. The last block has no C's, so its other k letters are A's. The remaining $5 - k$ A's are then in the middle block, and the rest of the middle block is k C's, which are exactly the C's not in the first block. So the blocks hold the following numbers of letters.

Block	A's	B's	C's
first	0	k	$5-k$
middle	$5-k$	0	k
last	k	$5-k$	0

Conversely, for each k, any three blocks with these letters make an arrangement with 5 of each letter that satisfies the conditions.

Each block is a row of letters of two kinds, so it is determined by the positions of one kind. The first block can be filled in $\binom{5}{k}$ ways, by choosing the positions of its B's, and the last block in $\binom{5}{k}$ ways, by choosing the positions of its A's. The middle block can be filled in $\binom{5}{5-k}$ ways, by choosing the positions of its A's, and by symmetry $\binom{5}{5-k} = \binom{5}{k}$. By the Multiplication Principle there are $\binom{5}{k}^3$ arrangements with k B's in the first block.

Each arrangement has exactly one value of k, so by the Addition Principle the number of arrangements is

$$\sum_{k=0}^{5} \binom{5}{k}^3 = 1 + 125 + 1000 + 1000 + 125 + 1 = 2252.$$

Answer: (A) $\displaystyle\sum_{k=0}^{5} \binom{5}{k}^3$. □

12. (1976 AHSME, Problem #23) For integers k and n such that $1 \le k < n$, let

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Then $\left(\frac{n-2k-1}{k+1}\right)\binom{n}{k}$ is an integer

(A) for all k and n **(B)** for all even values of k and n, but not for all k and n
(C) for all odd values of k and n, but not for all k and n
(D) if $k = 1$ or $n - 1$, but not for all odd values of k and n
(E) if n is divisible by k, but not for all even values of k and n

Strategy. Split the fraction as $\frac{n-k}{k+1} - 1$. The factor $\frac{n-k}{k+1}$ turns an entry of row n of Pascal's triangle into the next entry of the row.

Solution. Since $n - 2k - 1 = (n - k) - (k + 1)$,

$$\frac{n-2k-1}{k+1} = \frac{n-k}{k+1} - 1,$$

so

$$\left(\frac{n-2k-1}{k+1}\right)\binom{n}{k} = \binom{n}{k} \cdot \frac{n-k}{k+1} - \binom{n}{k}.$$

Neighboring entries of row n of Pascal's triangle satisfy $\binom{n}{k+1} = \binom{n}{k} \cdot \frac{n-k}{k+1}$ for $0 \le k \le n-1$, and here $1 \le k \le n-1$. So the expression equals

$$\binom{n}{k+1} - \binom{n}{k},$$

a difference of two numbers of subsets, which is an integer. This holds for all k and n with $1 \le k < n$, so (A) is correct; each of the other choices says that the expression is not an integer for some k and n.

Answer: (A) for all k and n. □

13. (2006 AIME II, Problem #4) Let $(a_1, a_2, a_3, \ldots, a_{12})$ be a permutation of $(1, 2, 3, \ldots, 12)$ for which

$$a_1 > a_2 > a_3 > a_4 > a_5 > a_6$$

and

$$a_6 < a_7 < a_8 < a_9 < a_{10} < a_{11} < a_{12}.$$

An example of such a permutation is $(6, 5, 4, 3, 2, 1, 7, 8, 9, 10, 11, 12)$. Find the number of such permutations.

Strategy. The entry a_6 is smaller than all the others. After that, the permutation is determined by the set of its first five entries.

Solution. By the first chain of inequalities a_6 is smaller than $a_1, \ldots, a_5$, and by the second it is smaller than $a_7, \ldots, a_{12}$. So a_6 is the smallest of the twelve numbers, and $a_6 = 1$. The first five entries form a set S of 5 of the eleven numbers $2, 3, \ldots, 12$, and they are the elements of S in decreasing order. The last six entries are the other 6 numbers, in increasing order. So the permutation is determined by S, and different sets S give different permutations.

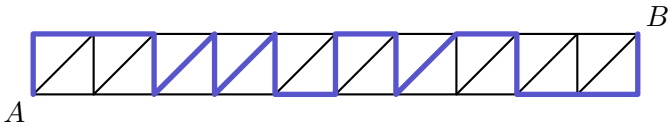
Conversely, for any 5-element subset S of $\{2, 3, \ldots, 12\}$, write the elements of S in decreasing order, then 1, then the other six numbers in increasing order. Both chains of inequalities hold, because 1 is smaller than every other number. So the permutations correspond to the 5-element subsets of an 11-element set, and there are

$$\binom{11}{5} = \frac{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 462$$

of them. The example in the problem comes from $S = \{2, 3, 4, 5, 6\}$.

Answer: 462. □

14. (2026 AIME II, Problem #2) The figure below shows a grid of 10 squares in a row. Each square has a diagonal connecting its lower left vertex to its upper right vertex. A bug moves along the line segments from vertex to vertex, never traversing the same segment twice and never moving from right to left along a horizontal or diagonal segment. Let N be the number of paths the bug can take from the lower left corner (A) to the upper right corner (B). One such path from A to B is shown by the thick line segments in the figure. Find $\sqrt{N}$.



Strategy. The bug never moves left, so it reaches the vertical segments one at a time, from left to right. Count point by point, one vertical segment at a time.

Solution. Number the vertical segments $0, 1, \ldots, 10$ from left to right, so that A is the bottom of segment 0 and B is the top of segment 10. Every horizontal or diagonal segment joins segment i to segment $i+1$ for some i, and the bug crosses it only from left to right. So the bug reaches the vertical segments in the order $0, 1, \ldots, 10$, and it uses each horizontal or diagonal segment at most once. After arriving at segment i, at its bottom or its top, the bug may travel along segment i, but at most once, since no segment may be traversed twice, and then it leaves to the right.

For $0 \le i \le 10$, let b_i and t_i be the numbers of ways for the bug to travel from A to the bottom and to the top of segment i, arriving from the left. At segment 0 the bug simply starts at A, so $b_0 = 1$ and $t_0 = 0$. For $i \le 9$, the bug is at the bottom of segment i, ready to leave, either right after arriving there or after arriving at the top and going down, so this happens in $b_i + t_i$ ways; in the same way, it is ready to leave from the top in $b_i + t_i$ ways. The bottom of segment $i + 1$ is reached only along a bottom side, from the bottom of segment i. The top of segment $i+1$ is reached along a top side, from the top of segment i, or along a diagonal, from the bottom of segment i. Counting point by point,

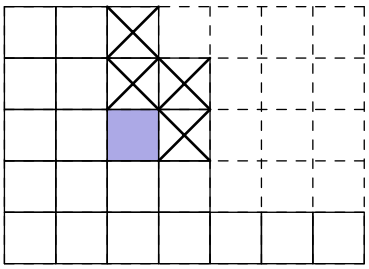
$$b_{i+1} = b_i + t_i \qquad \text{and} \qquad t_{i+1} = 2(b_i + t_i),$$

so $b_{i+1} + t_{i+1} = 3(b_i + t_i)$. Starting from $b_0 + t_0 = 1$, this gives $b_{10} + t_{10} = 3^{10}$.

Finally, each arrival at segment 10 ends at B in exactly one way: an arrival at the top is at B, and from the bottom the bug must go up segment 10 to B. So $N = 3^{10}$, and $\sqrt{N} = 3^5 = 243$.

Answer: 243. □

15. (1992 AIME, Problem #12) In a game of Chomp, two players alternately take bites from a 5-by-7 grid of unit squares. To take a bite, a player chooses one of the remaining squares, then removes ("eats") all squares in the quadrant defined by the left edge (extended upward) and the lower edge (extended rightward) of the chosen square. For example, the bite determined by the shaded square in the diagram would remove the shaded square and the four squares marked by ×. (The squares with two or more dotted edges have been removed from the original board in previous moves.)



The object of the game is to make one's opponent take the last bite. The diagram shows one of the many subsets of the set of 35 unit squares that can occur during the game of Chomp. How many different subsets are there in all? Include the full board and empty board in your count.

Strategy. Show that the remaining squares always form a staircase: each column is a stack of squares at the bottom, and the columns never get taller from left to right. Such a set is determined by its boundary, a path of 7 steps to the right and 5 steps down.

Solution. Number the columns 1 to 7 from left to right and the rows 1 to 5 from bottom to top. A bite at the square in column c and row r removes every remaining square that is in one of the columns $c, c+1, \ldots, 7$ and in one of the rows $r, r+1, \ldots, 5$.

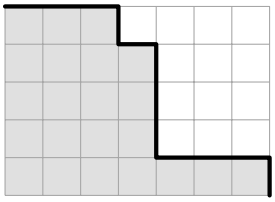
Call a set of squares a **staircase** if each column consists of its bottom few squares and the heights $h_1, h_2, \ldots, h_7$ of the columns never increase from left to right, that is,

$$5 \geq h_1 \geq h_2 \geq \cdots \geq h_7 \geq 0.$$

The full board is a staircase with every height 5. In a staircase, a bite at the square in column c and row r lowers each of the heights $h_c, \ldots, h_7$ that is greater than $r-1$ to $r-1$ and leaves the other heights unchanged. So each column still consists of its bottom squares, the heights from column c on stay in order, and h_{c-1} is still at least the new height of column c: a staircase stays a staircase. Hence every subset that occurs during the game is a staircase.

Conversely, every staircase occurs. Starting from the full board, go through the columns $j = 1, 2, \ldots, 7$ in turn. When column j is reached, it is at least h_j tall, since the bites so far have lowered it only to heights $h_i \geq h_j$ with $i < j$. If it is taller than h_j, take a bite at the square in column j and row $h_j + 1$. This makes column j exactly h_j tall, lowers the columns to its right to at most h_j, and leaves the columns to its left alone. At the end, every column j has height h_j.

The boundary between a staircase and the eaten squares runs along grid lines from the top left corner of the board to the bottom right corner: down the left edge to height h_1, right across column 1, down to height h_2, right across column 2, and so on, and after crossing column 7 at height h_7, down the right edge to the bottom. For the subset in the diagram the heights are $5, 5, 5, 4, 1, 1, 1$, and the boundary is the thick path below.



The boundary is a row of 7 steps to the right (R) and 5 steps down (D). Different staircases give different boundaries, and every row of 7 R's and 5 D's is the boundary of a staircase: the jth R runs at height h_j, which is 5 minus the number of D's before it, so the heights never increase. Replacing each D by a step up (U) turns the boundary into a lattice path from $(0, 0)$ to $(7, 5)$, so the number of subsets is

$$\binom{12}{5} = 792.$$

This count includes the full board, with every height 5, and the empty board, with every height 0.

Answer: 792. □

2. Answer Key

Problem	Letter	Value
1	C	9
2	A	4850
3	D	20
4	A	$15/x$
5	D	8
6	D	26
7	B	$4/5$
8	B	20
9		968
10		13
11	A	$\sum_{k=0}^{5} \binom{5}{k}^3$
12	A	for all k and n
13		462
14		243
15		792