

COUNTING PRINCIPLES AND PERMUTATIONS FOR AMC 10: SOLUTIONS

ANDY JO

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1. SOLUTIONS

Beginner.

1. (2020 AMC 10B, Problem #5) How many distinguishable arrangements are there of 1 brown tile, 1 purple tile, 2 green tiles, and 3 yellow tiles in a row from left to right? (Tiles of the same color are indistinguishable.)
 (A) 210 (B) 420 (C) 630 (D) 840 (E) 1050

Strategy. Tiles of the same color are identical, so this asks for the number of arrangements of a multiset.

Solution. There are $1 + 1 + 2 + 3 = 7$ tiles: one brown, one purple, two green and three yellow. By the formula for arrangements of a multiset, the number of distinguishable arrangements is

$$\frac{7!}{1!1!2!3!} = \frac{5040}{12} = 420.$$

Answer: (B) 420. □

2. (2003 AMC 10B, Problem #10) Nebraska, the home of the AMC, changed its license plate scheme. Each old license plate consisted of a letter followed by four digits. Each new license plate consists of three letters followed by three digits. By how many times is the number of possible license plates increased?
 (A) $\frac{26}{10}$ (B) $\frac{26^2}{10^2}$ (C) $\frac{26^2}{10}$ (D) $\frac{26^3}{10^3}$ (E) $\frac{26^3}{10^2}$

Strategy. Count the plates in each scheme with the Multiplication Principle, then divide.

Solution. Each character of a plate is chosen independently: 26 choices for a letter and 10 for a digit. An old plate has one letter and four digits, so there are $26 \cdot 10^4$ old plates. A new plate has three letters and three digits, so there are $26^3 \cdot 10^3$ new plates. The number of possible plates is multiplied by

$$\frac{26^3 \cdot 10^3}{26 \cdot 10^4} = \frac{26^2}{10}.$$

Answer: (C) $\frac{26^2}{10}$. □

3. (2009 AMC 10B, Problem #11) How many 7-digit palindromes (numbers that read the same backward as forward) can be formed using the digits 2, 2, 3, 3, 5, 5, 5?
 (A) 6 (B) 12 (C) 24 (D) 36 (E) 48

Strategy. In a 7-digit palindrome the middle digit stands alone and the other six digits form three matching pairs. Find the middle digit first.

Solution. In a 7-digit palindrome the first and seventh digits are equal, as are the second and sixth, and the third and fifth. So the six digits other than the middle one come in three pairs of equal digits, and each digit value appears an even number of times among them. The digit 5 appears three times, an odd number, so the middle digit must be 5. The other six digits 2, 2, 3, 3, 5, 5 then form the pairs: one pair of 2's, one pair of 3's and one pair of 5's.

The palindrome is determined by its first three digits, which are 2, 3 and 5 in some order. There are $3! = 6$ such orders, so there are 6 palindromes.

Answer: (A) 6. □

4. (2014 AMC 10A, Problem #4) Walking down Jane Street, Ralph passed four houses in a row, each painted a different color. He passed the orange house before the red house, and he passed the blue house before the yellow house. The blue house was not next to the yellow house. How many orderings of the colored houses are possible?
- (A) 2 (B) 3 (C) 4 (D) 5 (E) 6

Strategy. The blue and yellow houses have two conditions, so place them first. The orange and red houses then have only one possible order.

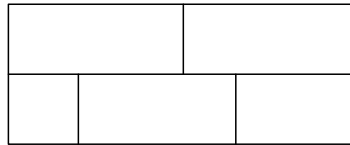
Solution. Number the positions 1 to 4 in the order Ralph passes the houses. The blue house comes before the yellow house and is not next to it, so the blue and yellow houses are in positions 1 and 3, 1 and 4, or 2 and 4. In each case the orange and red houses fill the other two positions with orange first, which can be done in exactly one way. Writing B, O, Y and R for blue, orange, yellow and red, the orderings are

BOYR, BORY, OBRY.

So there are 3 orderings.

Answer: (B) 3. □

5. (2022 AMC 10A, Problem #9) A rectangle is partitioned into 5 regions as shown. Each region is to be painted a solid color—red, orange, yellow, blue, or green—so that regions that touch are painted different colors, and colors can be used more than once. How many different colorings are possible?



- (A) 120 (B) 270 (C) 360 (D) 540 (E) 720

Strategy. Color the region that touches the most other regions first, then color the regions next to it.

Solution. Call the regions top left, top right, bottom left, bottom middle and bottom right. The bottom middle region touches all four other regions. The top left region touches the top right, bottom left and bottom middle regions, and the top right region touches the top left, bottom middle and bottom right regions. The bottom left region touches only the top left and bottom middle regions, and the bottom right region touches only the top right and bottom middle regions.

Color the bottom middle region first, in 5 ways. The top left region must have a different color, so it has 4 choices, and the top right region must differ from both of these, so it has 3 choices. The bottom left region must differ from the top left and bottom middle regions, which have different colors, so it has 3 choices. In the same way, the bottom right region must differ from the top right and bottom middle regions, so it has 3 choices. By the Multiplication Principle there are

$$5 \cdot 4 \cdot 3 \cdot 3 \cdot 3 = 540$$

colorings.

Answer: (D) 540. □

Intermediate.

6. (2011 AMC 10A, Problem #13) How many even integers are there between 200 and 700 whose digits are all different and come from the set $\{1, 2, 5, 7, 8, 9\}$?
 (A) 12 (B) 20 (C) 72 (D) 120 (E) 200

Strategy. The hundreds digit and the units digit are restricted, so choose them first and the tens digit last.

Solution. Every such integer has three digits. Its hundreds digit is from the set and the integer is between 200 and 700, so the hundreds digit is 2 or 5. The integer is even, so its units digit is 2 or 8.

If the hundreds digit is 2, the units digit must be 8, since the digits are different. If the hundreds digit is 5, the units digit can be 2 or 8. So there are $1 + 2 = 3$ ways to choose the hundreds and units digits. In each case the tens digit can be any of the 4 digits of the set not yet used. By the Multiplication Principle there are $3 \cdot 4 = 12$ integers.

Answer: (A) 12. □

7. (1988 AHSME, Problem #18) At the end of a professional bowling tournament, the top 5 bowlers have a play-off. First #5 bowls #4. The loser receives 5th prize and the winner bowls #3 in another game. The loser of this game receives 4th prize and the winner bowls #2. The loser of this game receives 3rd prize and the winner bowls #1. The winner of this game gets 1st prize and the loser gets 2nd prize. In how many orders can bowlers #1 through #5 receive the prizes?
 (A) 10 (B) 16 (C) 24 (D) 120 (E) none of these

Strategy. Each game has two possible results. Count the sequences of results, and check that different sequences give different prize orders.

Solution. Four games are played, and each has 2 possible winners, so by the Multiplication Principle there are $2^4 = 16$ possible sequences of results. The sequence of results determines who receives each prize.

Different sequences give different prize orders. Suppose two sequences first differ in some game. The earlier games have the same results, so the same two bowlers play this game in both sequences, but it has different losers. The prize given to the loser of this game therefore goes to different bowlers. Hence there are 16 orders.

Answer: (B) 16. □

8. (2020 AMC 10A, Problem #18) Let (a, b, c, d) be an ordered quadruple of not necessarily distinct integers, each one of them in the set $\{0, 1, 2, 3\}$. For how many such quadruples is it true that $a \cdot d - b \cdot c$ is odd? (For example, $(0, 3, 1, 1)$ is one such quadruple, because $0 \cdot 1 - 3 \cdot 1 = -3$ is odd.)
 (A) 48 (B) 64 (C) 96 (D) 128 (E) 192

Strategy. The difference $a \cdot d - b \cdot c$ is odd exactly when one of the two products is odd and the other is even. Count the pairs that give an odd product, then use cases.

Solution. The set $\{0, 1, 2, 3\}$ contains two odd numbers and two even numbers. The product $a \cdot d$ is odd exactly when a and d are both odd, which happens for $2 \cdot 2 = 4$ of the $4 \cdot 4 = 16$ pairs (a, d) . For the other 12 pairs, $a \cdot d$ is even. The same counts hold for $b \cdot c$.

The difference $a \cdot d - b \cdot c$ is odd exactly when one of the products is odd and the other is even. If $a \cdot d$ is odd and $b \cdot c$ is even, there are $4 \cdot 12 = 48$ quadruples. If $a \cdot d$ is even

and $b \cdot c$ is odd, there are $12 \cdot 4 = 48$ quadruples. These cases do not overlap, so by the Addition Principle there are $48 + 48 = 96$ quadruples.

Answer: (C) 96. □

9. (2005 AMC 10B, Problem #20) What is the average (mean) of all 5-digit numbers that can be formed by using each of the digits 1, 3, 5, 7, and 8 exactly once?

(A) 48000 (B) 49999.5 (C) 53332.8 (D) 55555 (E) 56432.8

Strategy. Count how many times each digit appears in each position over all the numbers.

Solution. The five digits can be arranged in $5! = 120$ ways, so there are 120 numbers. Fix a position, for example the units position, and a digit, for example 7. The numbers with 7 in that position are formed by arranging the other four digits in the other four positions, in $4! = 24$ ways. So each digit appears in each position exactly 24 times.

The units digits of the 120 numbers therefore add up to $24(1+3+5+7+8) = 24 \cdot 24 = 576$, and the digits in every other position also add up to 576. The sum of all the numbers is

$$576 \cdot (10000 + 1000 + 100 + 10 + 1) = 576 \cdot 11111,$$

so their average is

$$\frac{576 \cdot 11111}{120} = 4.8 \cdot 11111 = 53332.8.$$

Answer: (C) 53332.8. □

10. (2022 AMC 12A, Problem #10) What is the number of ways the numbers from 1 to 14 can be split into 7 pairs such that for each pair, the greater number is at least 2 times the smaller number?

(A) 108 (B) 120 (C) 126 (D) 132 (E) 144

Strategy. First show that each pair contains one number from 1 to 7 and one from 8 to 14. Then choose partners for 7, 6, ..., 1 in that order, starting with the most restricted number.

Solution. If both numbers in a pair were at least 8, the greater one would be at least $2 \cdot 8 = 16$, which is too large. So every pair contains a number from 1 to 7. There are 7 pairs and only 7 such numbers, so each pair contains exactly one number from 1 to 7, its smaller number, and one number from 8 to 14.

Choose the partners of the small numbers in the order 7, 6, 5, 4, 3, 2, 1.

- The partner of 7 must be at least 14, so it is 14: 1 way.
- The partner of 6 must be at least 12, and 14 is taken, so it is 12 or 13: 2 ways.
- The partner of 5 must be at least 10. Two of the numbers 10, 11, 12, 13, 14 are already taken, so there are 3 ways.
- The partner of 4 must be at least 8. Three of the seven large numbers are taken, so there are 4 ways.
- The partners of 3, 2 and 1 must be at least 6, 4 and 2, which every large number is.

So 3, 2 and 1 take the remaining large numbers in 3, 2 and 1 ways.

The number of options at each step does not depend on the earlier choices, so by the Multiplication Principle there are $1 \cdot 2 \cdot 3 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 144$ ways.

Answer: (E) 144. □

Advanced.

11. (1982 AHSME, Problem #11) How many integers with four different digits are there between 1,000 and 9,999 such that the absolute value of the difference between the first digit and the last digit is 2?

(A) 672 (B) 784 (C) 840 (D) 896 (E) 1,008

Strategy. The first and last digits are restricted, so choose them first. The two middle digits only need to differ from the other digits.

Solution. Let the first digit be a and the last digit be d . Then a is one of $1, 2, \dots, 9$, and $d = a - 2$ or $d = a + 2$, whichever of these is a digit.

- If $a = 1$, then $d = 3$: 1 pair.
- If a is one of $2, 3, 4, 5, 6, 7$, both $a - 2$ and $a + 2$ are digits: $6 \cdot 2 = 12$ pairs. For $a = 2$ the last digit can be 0, since only the first digit must be nonzero.
- If $a = 8$ or $a = 9$, only $d = a - 2$ is a digit: 2 pairs.

So there are $1 + 12 + 2 = 15$ ways to choose the first and last digits. The second digit can be any of the 8 digits different from a and d , and the third digit any of the 7 digits different from the other three. By the Multiplication Principle there are $15 \cdot 8 \cdot 7 = 840$ integers.

Answer: (C) 840. □

12. (2017 AMC 10A, Problem #19) Alice refuses to sit next to either Bob or Carla. Derek refuses to sit next to Eric. How many ways are there for the five of them to sit in a row of 5 chairs under these conditions?

(A) 12 (B) 16 (C) 28 (D) 32 (E) 40

Strategy. Alice has the most restrictions, so place her first and use casework on her chair.

Solution. Number the chairs 1 to 5 from left to right. Alice cannot sit next to Bob or Carla, so each of her neighbors is Derek or Eric.

Case 1: Alice in chair 1. Chair 2 holds Derek or Eric: 2 ways. The other one of Derek and Eric cannot sit in chair 3, next to chair 2, so this person sits in chair 4 or 5: 2 ways. Bob and Carla fill the last two chairs in 2 ways. This gives $2 \cdot 2 \cdot 2 = 8$ seatings, and by symmetry Alice in chair 5 gives 8 more.

Case 2: Alice in chair 2. Chairs 1 and 3 hold Derek and Eric in 2 orders, and these chairs are not next to each other. Bob and Carla fill chairs 4 and 5 in 2 ways. This gives 4 seatings, and by symmetry Alice in chair 4 gives 4 more.

Case 3: Alice in chair 3. Chairs 2 and 4 hold Derek and Eric in 2 orders, and these chairs are not next to each other. Bob and Carla fill chairs 1 and 5 in 2 ways. This gives 4 seatings.

By the Addition Principle there are $16 + 8 + 4 = 28$ seatings.

Answer: (C) 28. □

13. (2024 AMC 10B, Problem #17) In a race among 5 snails, there is at most one tie, but that tie can involve any number of snails. For example, the result of the race might be that Dazzler is first; Abby, Cyrus, and Elroy are tied for second; and Bruna is fifth. How many different results of the race are possible?

(A) 180 (B) 361 (C) 420 (D) 431 (E) 720

Strategy. The tied snails act like one block whose internal order does not matter. Line up the snails with the tied ones as a block, then divide by the number of orders inside the block.

Solution. If there is no tie, a result is an ordering of the 5 snails, so there are $5! = 120$ such results.

Suppose k snails tie, where $2 \leq k \leq 5$. Write a result as a row of the 5 snails from first place to last, with the tied snails next to each other as a block. In the example, the row is Dazzler, then Abby, Cyrus and Elroy as a block, then Bruna. The block fills k consecutive places of the row and can start at place $1, 2, \dots, 6 - k$, so it has $6 - k$ possible positions. The snails can fill the row in $5!$ ways, so there are $5!(6 - k)$ rows with a marked block. Each result appears $k!$ times among these rows, once for each order of the tied snails inside the block. So the number of results with a k -way tie is

$$\frac{5!(6 - k)}{k!} = \frac{120(6 - k)}{k!}.$$

For $k = 2, 3, 4, 5$ this gives

$$\frac{120 \cdot 4}{2} = 240, \quad \frac{120 \cdot 3}{6} = 60, \quad \frac{120 \cdot 2}{24} = 10, \quad \frac{120 \cdot 1}{120} = 1$$

results. By the Addition Principle there are

$$120 + 240 + 60 + 10 + 1 = 431$$

results.

Answer: (D) 431. □

14. (2018 AMC 10B, Problem #18) Three young brother-sister pairs from different families need to take a trip in a van. These six children will occupy the second and third rows in the van, each of which has three seats. To avoid disruptions, siblings may not sit right next to each other in the same row, and no child may sit directly in front of his or her sibling. How many seating arrangements are possible for this trip?
- (A) 60 (B) 72 (C) 92 (D) 96 (E) 120

Strategy. The middle seat of the second row has the most neighbors: two beside it and one behind it. Seat that child first, then the sibling, and then see what the conditions force in the remaining seats.

Solution. Call the seats of the second row L_2, M_2, R_2 from left to right, and the seats of the third row L_3, M_3, R_3 , so that each third-row seat is directly behind the second-row seat with the same letter.

Any of the 6 children can sit in M_2 . That child's sibling cannot sit in L_2 or R_2 , next to M_2 , or in M_3 , directly behind M_2 , so the sibling sits in L_3 or R_3 : 2 ways. The two cases are mirror images of each other, so suppose the sibling sits in L_3 .

The other four children, two brother-sister pairs, fill L_2, R_2, M_3 and R_3 . The child in R_3 sits next to M_3 and directly behind R_2 , so the children in M_3 and R_2 are not siblings of the child in R_3 . Therefore the sibling of the child in R_3 sits in L_2 , and the other pair fills M_3 and R_2 . The child in R_3 can be any of the 4 children, its sibling then goes to L_2 , and the other pair fills M_3 and R_2 in 2 ways. These seatings satisfy every condition, since M_3 and R_2 are neither in the same row nor one directly in front of the other, and neither are L_2 and R_3 .

By the Multiplication Principle there are $6 \cdot 2 \cdot 4 \cdot 2 = 96$ seating arrangements.

Answer: (D) 96. □

2. ANSWER KEY

Problem	Letter	Value
1	B	420
2	C	$26^2/10$
3	A	6
4	B	3
5	D	540
6	A	12
7	B	16
8	C	96
9	C	53332.8
10	E	144
11	C	840
12	C	28
13	D	431
14	D	96