# COUNTING PRINCIPLES AND PERMUTATIONS FOR AMC 10: PRACTICE SET SOLUTIONS

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## 1. Solutions

**Beginner.**

**1.** The six letters of the word COFFEE are arranged in a row. In how many distinct arrangements are the two F's next to each other?

*Strategy.* Glue the two F's into one block and arrange the block together with the other four letters. The two F's are identical, so the block has only one inner order.

*Solution.* The letters of COFFEE are C, O, two F's and two E's. Glue the two F's into a single block FF. Each arrangement with the two F's next to each other then corresponds to exactly one arrangement of the five units

$$\text{C}, \quad \text{O}, \quad \text{FF}, \quad \text{E}, \quad \text{E}$$

in a row. These units form a multiset in which E appears twice and each other unit once, so by the formula for arrangements of a multiset they have

$$\frac{5!}{2!} = \frac{120}{2} = 60$$

distinct arrangements. There is no extra factor for the order inside the block: its two letters are both F, so swapping them gives the same arrangement.

**Answer:** 60. □

**2.** (2022 AMC 10B, Problem #9) The sum

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{2021}{2022!}$$

can be expressed as $a - \dfrac{1}{b!}$, where $a$ and $b$ are positive integers. What is $a + b$?

**(A)** 2020 **(B)** 2021 **(C)** 2022 **(D)** 2023 **(E)** 2024

*Strategy.* Use $(k+1)! = (k+1) \cdot k!$ to write each term as the difference of the reciprocals of two consecutive factorials. Then the pieces of neighboring terms cancel, and only the first and last pieces remain.

*Solution.* For each positive integer $k$, write the numerator $k$ as $(k+1)-1$. Since $(k+1)! = (k+1) \cdot k!$,

$$\frac{k}{(k+1)!} = \frac{k+1}{(k+1)!} - \frac{1}{(k+1)!} = \frac{1}{k!} - \frac{1}{(k+1)!}.$$

Applying this to the terms with $k = 1, 2, \ldots, 2021$ turns the sum into

$$\left(\frac{1}{1!} - \frac{1}{2!}\right) + \left(\frac{1}{2!} - \frac{1}{3!}\right) + \cdots + \left(\frac{1}{2021!} - \frac{1}{2022!}\right).$$

The second piece of each bracket cancels the first piece of the next one, so the sum equals

$$\frac{1}{1!} - \frac{1}{2022!} = 1 - \frac{1}{2022!}.$$

This is the only way to write the sum in the given form. For a positive integer $b$, $0 < \frac{1}{b!} \leq 1$, so $a - 1 \leq a - \frac{1}{b!}$. The sum is less than 1, so $a - 1 < 1$, and the positive integer $a$ must be 1. Then $\frac{1}{b!} = \frac{1}{2022!}$, so $b = 2022$ and $a + b = 2023$.

**Answer:** (D) 2023. □

**3.** (2003 AMC 10B, Problem #13) Let $\clubsuit(x)$ denote the sum of the digits of the positive integer $x$. For example, $\clubsuit(8) = 8$ and $\clubsuit(123) = 1 + 2 + 3 = 6$. For how many two-digit values of $x$ is $\clubsuit(\clubsuit(x)) = 3$?

**(A)** 3 **(B)** 4 **(C)** 6 **(D)** 9 **(E)** 10

*Strategy.* The digit sum of a two-digit number is at most 18. Find which values up to 18 have digit sum 3, then count the two-digit numbers for each value separately and add.

*Solution.* A two-digit number has a tens digit from 1 to 9 and a units digit from 0 to 9, so its digit sum $\clubsuit(x)$ is between 1 and 18. Among the integers from 1 to 18, only 3 and 12 have digit sum 3. So $\clubsuit(\clubsuit(x)) = 3$ exactly when $\clubsuit(x) = 3$ or $\clubsuit(x) = 12$. These two cases do not overlap, so by the Addition Principle the answer is the sum of their counts.

**Case 1: $\clubsuit(x) = 3$.** The tens digit is 1, 2 or 3, and the units digit is 3 minus the tens digit. This gives the 3 numbers 12, 21 and 30.

**Case 2: $\clubsuit(x) = 12$.** The units digit is at most 9, so the tens digit is at least 3, and each tens digit from 3 to 9 gives exactly one number. This gives the 7 numbers 39, 48, 57, 66, 75, 84 and 93.

In all there are $3 + 7 = 10$ values of $x$.

**Answer:** (E) 10. □

**4.** (2003 AMC 10B, Problem #16) A restaurant offers three desserts, and exactly twice as many appetizers as main courses. A dinner consists of an appetizer, a main course, and a dessert. What is the least number of main courses that the restaurant should offer so that a customer could have a different dinner each night in the year 2003?

**(A)** 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

*Strategy.* A dinner is a sequence of three choices, so the Multiplication Principle gives the number of dinners in terms of the number of main courses. Compare this number with the number of nights in 2003.

*Solution.* Suppose the restaurant offers $m$ main courses. Then it offers $2m$ appetizers and 3 desserts. A dinner is made by choosing an appetizer, a main course and a dessert, so by the Multiplication Principle there are

$$2m \cdot m \cdot 3 = 6m^2$$

different dinners. The number 2003 is not a multiple of 4, so 2003 is not a leap year and has 365 nights. A different dinner every night requires $6m^2 \geq 365$. The value of $6m^2$ grows with $m$, and

$$6 \cdot 7^2 = 294 < 365 \leq 384 = 6 \cdot 8^2,$$

so the least number of main courses is 8.

**Answer:** (E) 8. □

**5.** How many distinct arrangements of the letters in the word PARADE have no two vowels next to each other?

*Strategy.* Arrange the consonants first, then put the vowels into different gaps between and around them. Label the two A's while placing the vowels, then divide by the number of ways to order the labels.

*Solution.* The word PARADE has the three consonants P, R, D, which are all different, and the three vowels A, A, E.

The consonants can be arranged in $3! = 6$ orders. Each order has 4 gaps: one before the first consonant, two between neighboring consonants and one after the last consonant.

No two vowels are next to each other exactly when the three vowels go into three different gaps.

Label the A's for the moment as $A_1$ and $A_2$. The three different vowels $A_1$, $A_2$, E can be put into three different gaps in $P(4,3) = 4 \cdot 3 \cdot 2 = 24$ ways. Once the labels are removed, each placement of the vowels has been counted twice, once for each order of $A_1$ and $A_2$, so the vowels can be placed in $24/2 = 12$ ways.

By the Multiplication Principle there are $6 \cdot 12 = 72$ arrangements.

**Answer:** 72. □

**Intermediate.**

**6.** (2021 Fall AMC 10A, Problem #18) A farmer's rectangular field is partitioned into a 2 by 2 grid of 4 rectangular sections as shown in the figure. In each section the farmer will plant one crop: corn, wheat, soybeans, or potatoes. The farmer does not want to grow corn and wheat in any two sections that share a border, and the farmer does not want to grow soybeans and potatoes in any two sections that share a border. Given these restrictions, in how many ways can the farmer choose crops to plant in each of the four sections of the field?



**(A)** 12 **(B)** 64 **(C)** 84 **(D)** 90 **(E)** 144

*Strategy.* Each crop has exactly one crop that may not border it. Plant the two sections that share no border first, then count the choices for the other two sections, in cases according to whether the first two crops are the same.

*Solution.* Call the sections top left, top right, bottom left and bottom right. Two sections share a border when they are in the same row or the same column. So the top left and bottom right sections share no border, and neither do the top right and bottom left sections, while each of the top right and bottom left sections borders both the top left and the bottom right sections.

Call corn and wheat partners, and also soybeans and potatoes. Every crop has exactly one partner, and the condition says that two sections that share a border never hold partners. The same crop may be planted in two sections that share a border.

Plant the top left and bottom right sections first. They share no border, so any two crops are allowed. The top right section must then avoid the partners of these two crops, and so must the bottom left section. These two sections share no border with each other, so they can be planted independently.

**Case 1: the top left and bottom right sections get the same crop.** There are 4 ways to choose it. The top right and bottom left sections must each avoid only the partner of this crop, so each has 3 choices. This case gives $4 \cdot 3 \cdot 3 = 36$ ways.

**Case 2: they get different crops.** There are $4 \cdot 3 = 12$ ways to choose them. Different crops have different partners, so the top right and bottom left sections must each avoid two crops, and each has 2 choices. This case gives $12 \cdot 2 \cdot 2 = 48$ ways.

The cases do not overlap, so by the Addition Principle there are $36 + 48 = 84$ ways.

**Answer:** (C) 84. □

**7.** (2019 AMC 12A, Problem #13) How many ways are there to paint each of the integers 2, 3, ..., 9 either red, green, or blue so that each number has a different color from each of its proper divisors?

**(A)** 144 **(B)** 216 **(C)** 256 **(D)** 384 **(E)** 432

*Strategy.* List the pairs of painted numbers in which one is a proper divisor of the other, and paint in an order that keeps the number of choices fixed. The number 6 has two painted proper divisors, 2 and 3, so its number of choices depends on whether they match, and it needs two cases.

*Solution.* The number 1 is not painted, so only the proper divisors from 2 to 9 matter. Among these, 4 has the proper divisor 2; 6 has 2 and 3; 8 has 2 and 4; 9 has 3. The numbers 2, 3, 5 and 7 have none.

The numbers 5 and 7 are not involved in any condition, so they can be painted in $3 \cdot 3 = 9$ ways.

Next paint 2, 3 and 6. The number 6 must differ from both 2 and 3.

- If 2 and 3 get the same color, which happens in 3 ways, then 6 has 2 choices.
- If 2 and 3 get different colors, which happens in $3 \cdot 2 = 6$ ways, then 6 has 1 choice.

So 2, 3 and 6 can be painted in $3 \cdot 2 + 6 \cdot 1 = 12$ ways.

The number 4 must differ from 2, so it has 2 choices. The number 8 must differ from 2 and from 4, which have different colors, so it has 1 choice. The number 9 must differ from 3, so it has 2 choices. These numbers of choices do not depend on the earlier choices, so by the Multiplication Principle the number of ways is

$$9 \cdot 12 \cdot 2 \cdot 1 \cdot 2 = 432.$$

**Answer:** (E) 432. □

**8.** (2003 AIME I, Problem #1) Given that $\dfrac{((3!)!)!}{3!} = k \cdot n!$, where $k$ and $n$ are positive integers and $n$ is as large as possible, find $k + n$.

*Strategy.* Compute $(3!)! = 720$ and simplify $720!/3!$ with the relation $n! = n \cdot (n-1)!$. Then check that $n$ cannot be 720 or more.

*Solution.* Since $3! = 6$ and $6! = 720$, the left side is $720!/6$. Using $720! = 720 \cdot 719!$,

$$\frac{720!}{6} = \frac{720 \cdot 719!}{6} = 120 \cdot 719!.$$

So $n = 719$ works, with $k = 120$. No larger $n$ works: if $n \geq 720$, then $k \cdot n! \geq n! \geq 720!$, which is greater than $720!/6$. Hence $n = 719$ and $k = 120$, and $k + n = 839$.

**Answer:** 839. □

**9.** (2003 AIME II, Problem #3) Define a *good word* as a sequence of letters that consists only of the letters $A$, $B$, and $C$ — some of these letters may not appear in the sequence — and in which $A$ is never immediately followed by $B$, $B$ is never immediately followed by $C$, and $C$ is never immediately followed by $A$. How many seven-letter good words are there?

*Strategy.* Build the word from left to right. At each position after the first, the conditions exclude exactly one letter, the one that may not follow the previous letter, so there are always 2 choices, although which letter is excluded varies.

*Solution.* Choose the seven letters of the word from left to right. The first letter can be any of $A$, $B$ and $C$, so it has 3 choices. Each later letter must not be the letter that is forbidden right after the letter before it: $B$ after $A$, $C$ after $B$, and $A$ after $C$. Which

letter is forbidden depends on the letter before it, but in every case exactly one of the three letters is forbidden, so each of the other six positions has exactly 2 choices. Different sequences of choices give different words, so by the Multiplication Principle there are

$$3 \cdot 2^6 = 192$$

seven-letter good words.

**Answer:** 192. □

**10.** A chef prepares seven different dishes, one at a time. Among them are bread, soup, stew, salad and dessert. The bread must be prepared before both the soup and the stew, and the salad must be prepared before the dessert. In how many orders can the chef prepare the seven dishes?

*Strategy.* Replace each group of dishes with a required relative order by copies of one placeholder: one symbol for the bread, soup and stew, and another for the salad and dessert. Then fill the placeholders from left to right in the ways the conditions allow.

*Solution.* Write X in place of each of the bread, the soup and the stew, write Y in place of each of the salad and the dessert, and call the other two dishes P and Q. An order of the seven dishes becomes an arrangement of the multiset X, X, X, Y, Y, P, Q, which has

$$\frac{7!}{3!\,2!} = \frac{5040}{12} = 420$$

distinct arrangements.

To turn such an arrangement into an order that meets the conditions, fill the three X's from left to right. The bread must come before both the soup and the stew, so it takes the first X, and the soup and the stew take the other two X's in either order: 2 ways. The salad must come before the dessert, so the salad takes the first Y and the dessert takes the second: 1 way. Every valid order arises in exactly one way from an arrangement and these choices, so by the Multiplication Principle the number of orders is

$$420 \cdot 2 = 840.$$

**Answer:** 840. □

**Advanced.**

**11.** (2008 AMC 10B, Problem #21) Ten chairs are evenly spaced around a round table and numbered clockwise from 1 through 10. Five married couples are to sit in the chairs with men and women alternating, and no one is to sit either next to or directly across from his or her spouse. How many seating arrangements are possible?

**(A)** 240 **(B)** 360 **(C)** 480 **(D)** 540 **(E)** 720

*Strategy.* Seat the men first. Then find where each woman may sit, and show that one woman's choice forces the choices of all the others.

*Solution.* The men and women alternate, so the men sit either in all the odd-numbered chairs or in all the even-numbered chairs, which gives 2 choices. The 5 men can be arranged in their 5 chairs in $5! = 120$ ways.

Now seat the women in the other five chairs. Chair numbers continue around the table, so chair 11 is chair 1 and chair 0 is chair 10. Consider a man in chair $c$. Of the five chairs left for the women, two are next to him (chairs $c-1$ and $c+1$) and one is directly across from him (chair $c+5$). So his wife must sit in one of the other two chairs, $c+3$ or $c-3$: three seats clockwise or three seats counterclockwise from her husband.

Suppose some wife sits three seats clockwise from her husband in chair $c$, that is, in chair $c+3$. Chair $c+3$ is also three seats counterclockwise from chair $c+6$, so the wife of the man in chair $c+6$ cannot sit there and must sit three seats clockwise from her husband as well. Repeating this argument moves from chair $c$ to chairs $c+6$, $c+12$, $c+18$, and $c+24$, which are the five men's chairs in some order. So every wife sits three seats clockwise from her husband. In the same way, if one wife sits three seats counterclockwise from her husband, all of them do. Both of these seatings satisfy the conditions, so there are exactly 2 ways to seat the women.

The total is $2 \cdot 120 \cdot 2 = 480$.

**Answer:** (C) 480. □

**12.** (2007 AIME II, Problem #1) A mathematical organization is producing a set of commemorative license plates. Each plate contains a sequence of five characters chosen from the four letters in AIME and the four digits in 2007. No character may appear in a sequence more times than it appears among the four letters in AIME or the four digits in 2007. A set of plates in which each possible sequence appears exactly once contains $N$ license plates. Find $\frac{N}{10}$.

*Strategy.* The characters are seven different symbols, and only 0 may appear twice. Split the sequences by whether they contain two 0's, and count each case with $P(n,k)$.

*Solution.* The characters are the seven different symbols A, I, M, E, 2, 0 and 7. In a sequence, 0 may appear at most twice and every other symbol at most once. Split the sequences into two cases, which do not overlap.

**Case 1: at most one 0.** The five characters are all different. They are 5 of the 7 symbols, chosen and arranged in order, so there are

$$P(7,5) = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = 2520$$

such sequences.

**Case 2: two 0's.** The other three characters are different and none of them is 0. First choose the places of the two 0's. Writing $*$ for a place that does not hold a 0, a choice of places is an arrangement of the multiset $0, 0, *, *, *$, so by the formula for arrangements of a multiset there are

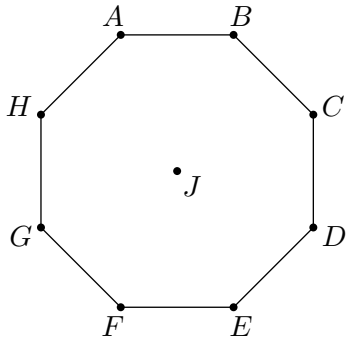
$$\frac{5!}{2!\,3!} = 10$$

choices. Then fill the three places marked $*$ from left to right with different symbols from the 6 symbols other than 0, in $P(6,3) = 6 \cdot 5 \cdot 4 = 120$ ways. This case gives $10 \cdot 120 = 1200$ sequences.

By the Addition Principle, $N = 2520 + 1200 = 3720$, so $\frac{N}{10} = 372$.

**Answer:** 372. □

**13.** (2013 AMC 10B, Problem #22) The regular octagon $ABCDEFGH$ has its center at $J$. Each of the vertices and the center are to be associated with one of the digits 1 through 9, with each digit used once, in such a way that the sums of the numbers on the lines $AJE$, $BJF$, $CJG$, and $DJH$ are all equal. In how many ways can this be done?




**(A)** 384 **(B)** 576 **(C)** 1152 **(D)** 1680 **(E)** 3456

*Strategy.* Each line is $J$ together with two opposite vertices. Adding up the four lines shows which digits can go at $J$. For each of them the pairs of opposite digits are forced, and placing the pairs on the lines is a permutation of four objects.

*Solution.* Let $j$ be the digit at $J$ and $s$ the common sum. Each line consists of $J$ and one of the pairs of opposite vertices $\{A, E\}$, $\{B, F\}$, $\{C, G\}$, $\{D, H\}$, so the two digits of each pair add up to $s - j$. The four pairs use the eight digits other than $j$, whose sum is $45 - j$, so $4(s - j) = 45 - j$. Hence $45 - j$ is a multiple of 4. Among the digits 1 to 9 this holds only for $j = 1$, 5 and 9, which make the sum of each pair 11, 10 and 9.

Fix one of these values of $j$, and let $t = (45 - j)/4$ be the sum of each pair. A digit $d$ other than $j$ must be paired with $t - d$, which is also a digit other than $j$ and different from $d$. So the eight digits split into pairs in exactly one way:

- for $j = 1$: $\{2, 9\}$, $\{3, 8\}$, $\{4, 7\}$, $\{5, 6\}$;
- for $j = 5$: $\{1, 9\}$, $\{2, 8\}$, $\{3, 7\}$, $\{4, 6\}$;
- for $j = 9$: $\{1, 8\}$, $\{2, 7\}$, $\{3, 6\}$, $\{4, 5\}$.

The four pairs can be assigned to the four pairs of opposite vertices in $4! = 24$ ways, and within each pair of opposite vertices the two digits can be placed in 2 orders. Every such placement makes all four line sums equal to $j + t$. By the Multiplication Principle each value of $j$ gives $24 \cdot 2^4 = 384$ ways, and the three values of $j$ give $3 \cdot 384 = 1152$ ways in all.

**Answer:** (C) 1152. □

**14.** (2005 AIME I, Problem #5) Robert has 4 indistinguishable gold coins and 4 indistinguishable silver coins. Each coin has an engraving of a face on one side, but not on the other. He wants to stack the eight coins on a table into a single stack so that no two adjacent coins are face to face. Find the number of possible distinguishable arrangements of the 8 coins.

*Strategy.* Describe a stack by its sequence of colors and by which way each coin faces. The colors form an arrangement of a multiset, and the condition on the faces involves only the orientations, which can follow only a few patterns.

*Solution.* Read the stack from the bottom up. A stack is described by its sequence of colors and by whether each coin has its face up or down. Coins of the same color are identical, so two stacks are distinguishable exactly when these descriptions differ.

The colors form an arrangement of a multiset of four G's and four S's, for gold and silver, so by the formula for arrangements of a multiset there are $8!/(4!\,4!) = 70$ color sequences.

The faces of two neighboring coins touch precisely when the face of the lower one points up and the face of the upper one points down. So once a face-up coin appears, every coin above it is face up as well, and the orientations are some number of face-down coins with face-up coins above them. The number of face-down coins can be any of $0, 1, \ldots, 8$, which gives 9 patterns of orientations, and each of them meets the condition.

The condition involves only the orientations, so every color sequence can go with every pattern, and by the Multiplication Principle there are $70 \cdot 9 = 630$ arrangements.

**Answer:** 630. □

**15.** (2013 AMC 10A, Problem #24) Central High School is competing against Northern High School in a backgammon match. Each school has three players, and the contest rules require that each player play two games against each of the other school's players. The match takes place in six rounds, with three games played simultaneously in each round. In how many different ways can the match be scheduled?

**(A)** 540 **(B)** 600 **(C)** 720 **(D)** 810 **(E)** 900

*Strategy.* In each round, the three games pair each Central player with a different Northern player. Find which lists of six such pairings give every matchup exactly twice, then count the orders of the rounds as arrangements of a multiset.

*Solution.* Call the Central players $A$, $B$, $C$ and the Northern players $X$, $Y$, $Z$, and write $AX$ for a game between $A$ and $X$. In each round every player plays exactly one game, so a round pairs each of $A$, $B$, $C$ with a different one of $X$, $Y$, $Z$. There are $3! = 6$ such pairings, and they split into two groups of three:

$$\text{Group 1:}\quad \{AX, BY, CZ\},\quad \{AY, BZ, CX\},\quad \{AZ, BX, CY\};$$
$$\text{Group 2:}\quad \{AX, BZ, CY\},\quad \{AY, BX, CZ\},\quad \{AZ, BY, CX\}.$$

Within a group, each of the 9 matchups appears in exactly one pairing, and every pairing in Group 1 shares exactly one matchup with every pairing in Group 2.

The schedule uses each pairing some number of times, with 6 rounds in all, and each matchup must occur exactly twice. Take any pairing $P$ in Group 1 and any pairing $Q$ in Group 2, and let $M$ be the matchup they share. The only pairings that contain $M$ are $P$ and $Q$, so the number of times $P$ is used plus the number of times $Q$ is used is 2. Since this holds for every such $P$ and $Q$, the three pairings in Group 1 are each used the same number of times $u$, and the three in Group 2 are each used $2 - u$ times.

**Case $u = 1$.** Each of the six pairings is used once, and the rounds can be put in order in $6! = 720$ ways.

**Case $u = 0$ or $u = 2$.** The three pairings of one group are each used twice, and the other group is not used. The six rounds form a multiset of three pairings, each appearing twice, so they can be put in order in

$$\frac{6!}{2!\,2!\,2!} = 90$$

ways. There are 2 choices for the group, which gives 180 ways.

The total is $720 + 180 = 900$.

**Answer:** (E) 900. □

## 2. Answer Key

| Problem | Letter | Value |
|---|---|---|
| 1 | | 60 |
| 2 | D | 2023 |
| 3 | E | 10 |
| 4 | E | 8 |
| 5 | | 72 |
| 6 | C | 84 |
| 7 | E | 432 |
| 8 | | 839 |
| 9 | | 192 |
| 10 | | 840 |
| 11 | C | 480 |
| 12 | | 372 |
| 13 | C | 1152 |
| 14 | | 630 |
| 15 | E | 900 |