POLYNOMIALS, INEQUALITIES, AND NUMBER THEORY FOR AMC 10

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1. Polynomials

This section covers factoring, polynomial division, and the relations between the coefficients and the roots of a polynomial.

1.1. **Factoring.** To **factor** a polynomial is to write it as a product of simpler polynomials. The first step is to factor out the **greatest common factor** (GCF) of all the terms.

The following identities hold for all numbers a and b:

$$\begin{aligned} \text{Difference of squares:} \quad & a^2 - b^2 = (a-b)(a+b). \\ \text{Sum of cubes:} \quad & a^3 + b^3 = (a+b)(a^2 - ab + b^2). \\ \text{Difference of cubes:} \quad & a^3 - b^3 = (a-b)(a^2 + ab + b^2). \\ \text{Perfect square trinomials:} \quad & a^2 \pm 2ab + b^2 = (a \pm b)^2. \end{aligned}$$

A polynomial with four terms can often be factored by **grouping**: pair the terms that share a common factor, factor each pair, then factor out the common binomial. For example,

$$x^3 + 2x^2 + 3x + 6 = x^2(x+2) + 3(x+2) = (x+2)(x^2+3).$$

Some expressions match one of the identities only after a term is added and subtracted. Completing the square is the usual way to do this. For example,

$$a^4 + 4b^4 = (a^4 + 4a^2b^2 + 4b^4) - 4a^2b^2 = (a^2 + 2b^2)^2 - (2ab)^2,$$

and the difference of squares gives

$$a^4 + 4b^4 = (a^2 + 2b^2 - 2ab)(a^2 + 2b^2 + 2ab).$$

This is **Sophie Germain's identity**.

Example.

Problem 1.1.1 (2022 AMC 10B, Problem #13)**.** The positive difference between a pair of primes is equal to 2, and the positive difference between the cubes of the two primes is 31106. What is the sum of the digits of the least prime that is greater than those two primes?

(A) 8 **(B)** 10 **(C)** 11 **(D)** 13 **(E)** 16

Strategy. Call the primes p and $p+2$. The difference of cubes factors, and one factor is $(p+2)-p = 2$. The other factor is a quadratic in p, which completing the square solves.

Solution. Let the primes be p and $p+2$. By the difference of cubes,

$$(p+2)^3 - p^3 = \big((p+2) - p\big)\big((p+2)^2 + (p+2)p + p^2\big) = 2(3p^2 + 6p + 4).$$

So $2(3p^2 + 6p + 4) = 31106$, that is, $3p^2 + 6p + 4 = 15553$. Completing the square, $3p^2 + 6p + 4 = 3(p+1)^2 + 1$, so

$$3(p+1)^2 = 15552 \qquad \text{and} \qquad (p+1)^2 = 5184 = 72^2.$$

Since $p > 0$, $p + 1 = 72$, so the primes are 71 and 73. As a check, $73^3 - 71^3 = 389017 - 357911 = 31106$. Of the numbers after 73, the even numbers 74, 76, and 78 are composite, and so are $75 = 3 \cdot 25$ and $77 = 7 \cdot 11$, but 79 is prime. So the least prime greater than 73 is 79, and the sum of its digits is $7 + 9 = 16$.

Answer: (E) 16. □

1.2. **Polynomial Division, Remainders, and Roots.**

Theorem 1.2.1 (Division Algorithm)**.** *For polynomials $p(x)$ and $d(x)$ with $d(x)$ not the zero polynomial, there are unique polynomials $q(x)$ and $r(x)$ with*

$$p(x) = d(x)q(x) + r(x), \quad r(x) = 0 \text{ or } \deg r < \deg d.$$

The polynomials $q(x)$ and $r(x)$ are the ***quotient*** *and the* ***remainder****.*

Theorem 1.2.2 (Remainder Theorem)**.** *The remainder when $p(x)$ is divided by $x - a$ is $p(a)$.*

Theorem 1.2.3 (Factor Theorem)**.** *$x - a$ divides $p(x)$ if and only if $p(a) = 0$.*

The Remainder Theorem turns remainders upon division by $x-a$ into values of the polynomial, and the Division Algorithm gives the form of a remainder upon division by a polynomial of higher degree.

Example.

Problem 1.2.4 (1999 AHSME, Problem #17)**.** Let $P(x)$ be a polynomial such that when $P(x)$ is divided by $x - 19$, the remainder is 99, and when $P(x)$ is divided by $x - 99$, the remainder is 19. What is the remainder when $P(x)$ is divided by $(x - 19)(x - 99)$?

(A) $-x + 80$ **(B)** $x + 80$ **(C)** $-x + 118$ **(D)** $x + 118$ **(E)** 0

Strategy. By the Remainder Theorem, $P(19) = 99$ and $P(99) = 19$. The divisor has degree 2, so the remainder has degree at most 1. Write it as $mx + n$ and use the two known values to find m and n.

Solution. By the Division Algorithm,

$$P(x) = (x - 19)(x - 99)Q(x) + R(x)$$

for some polynomial $Q(x)$, where $R(x)$ has degree at most 1. Write $R(x) = mx + n$. By the Remainder Theorem, $P(19) = 99$ and $P(99) = 19$. At $x = 19$ and at $x = 99$ the first term is zero, so

$$P(19) = 19m + n = 99,$$
$$P(99) = 99m + n = 19.$$

Subtracting the first equation from the second gives $80m = -80$, so $m = -1$. Substituting $m = -1$ into the first equation gives $n = 118$. Therefore the remainder is $R(x) = -x + 118$.

Answer: (C) $-x + 118$. □

1.3. **Vieta's Formulas.** Vieta's formulas for a quadratic, $r_1 + r_2 = -\frac{b}{a}$ and $r_1 r_2 = \frac{c}{a}$ for the roots r_1, r_2 of $ax^2 + bx + c = 0$, extend to a cubic.

Theorem 1.3.1 (Vieta's Formulas for a Cubic)**.** *Let $a \neq 0$, and let r_1, r_2, r_3 be the roots of $ax^3 + bx^2 + cx + d = 0$, so that $ax^3 + bx^2 + cx + d = a(x - r_1)(x - r_2)(x - r_3)$. Then*

$$r_1 + r_2 + r_3 = -\frac{b}{a}, \quad r_1 r_2 + r_2 r_3 + r_3 r_1 = \frac{c}{a}, \quad r_1 r_2 r_3 = -\frac{d}{a}.$$

Proof. Expand $a(x - r_1)(x - r_2)(x - r_3)$ and compare coefficients. □

Vieta's formulas give the value of a polynomial in the roots that does not change when the roots are swapped, without finding the roots. For a cubic, squaring the sum of the roots gives

$$r_1^2 + r_2^2 + r_3^2 = (r_1 + r_2 + r_3)^2 - 2(r_1 r_2 + r_2 r_3 + r_3 r_1).$$

Example.

Problem 1.3.2 (1967 AHSME, Problem #35)**.** The roots of $64x^3 - 144x^2 + 92x - 15 = 0$ are in arithmetic progression. The difference between the largest and smallest roots is:

(A) 2 **(B)** 1 **(C)** $\frac{1}{2}$ **(D)** $\frac{3}{8}$ **(E)** $\frac{1}{4}$

Strategy. Write the roots as $m - t$, m, $m + t$ with $t \geq 0$. The sum of the roots gives m, and the sum of the products of pairs of roots then gives t.

Solution. Three numbers in arithmetic progression can be written as $m - t$, m, $m + t$ with $t \geq 0$. By Vieta's formulas, the sum of the roots is $\frac{144}{64} = \frac{9}{4}$, so $3m = \frac{9}{4}$ and $m = \frac{3}{4}$. The sum of the products of pairs of roots is $\frac{92}{64} = \frac{23}{16}$, so

$$(m - t)m + m(m + t) + (m - t)(m + t) = 3m^2 - t^2 = \frac{23}{16}.$$

With $m = \frac{3}{4}$, $3m^2 = \frac{27}{16}$, so $t^2 = \frac{27}{16} - \frac{23}{16} = \frac{1}{4}$ and $t = \frac{1}{2}$. The roots are $\frac{1}{4}$, $\frac{3}{4}$, and $\frac{5}{4}$. As a check, their product is $\frac{15}{64}$, and Vieta's formulas give $r_1r_2r_3 = -\frac{-15}{64} = \frac{15}{64}$. The difference between the largest and smallest roots is $(m + t) - (m - t) = 2t = 1$.

Answer: (B) 1. □

2. Inequalities

This section covers the Arithmetic Mean–Geometric Mean inequality (AM–GM), which is used to find the largest or smallest value of an expression.

2.1. The AM–GM Inequality.

Theorem 2.1.1 (AM–GM)**.** *For nonnegative real numbers $x_1, \ldots, x_n$,*

$$\frac{x_1 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n},$$

with equality if and only if $x_1 = \cdots = x_n$.

For $n = 2$ the inequality reads

$$\frac{x + y}{2} \geq \sqrt{xy} \qquad (x, y \geq 0).$$

It follows from $(\sqrt{x} - \sqrt{y})^2 \geq 0$, which expands to $x + y - 2\sqrt{xy} \geq 0$. Equality holds exactly when $\sqrt{x} = \sqrt{y}$.

AM–GM gives a constant lower bound for a sum when the product of the terms is constant, and a constant upper bound for a product when the sum of the factors is constant. In both cases the bound is attained only if all the terms can be made equal.

Examples.

Problem 2.1.2. Find the minimum value of

$$a + \frac{4}{a - 5}$$

over all real numbers $a > 5$.

Strategy. The terms a and $\frac{4}{a-5}$ do not have a constant product, but $a - 5$ and $\frac{4}{a-5}$ do. Rewrite the expression in terms of $x = a - 5$.

Solution. Let $x = a - 5$, so $x > 0$ and

$$a + \frac{4}{a-5} = x + \frac{4}{x} + 5.$$

By AM–GM on the positive numbers x and $\frac{4}{x}$,

$$\frac{x + \frac{4}{x}}{2} \geq \sqrt{x \cdot \frac{4}{x}} = 2, \qquad \text{so} \qquad x + \frac{4}{x} \geq 4.$$

Hence the expression is at least $4 + 5 = 9$. Equality holds when $x = \frac{4}{x}$, that is, $x = 2$ and $a = 7$, and then the expression equals $7 + \frac{4}{2} = 9$.

Answer: 9. □

Problem 2.1.3 (2000 AMC 12, Problem #12)**.** Let A, M, and C be nonnegative integers such that $A + M + C = 12$. What is the maximum value of

$$A \cdot M \cdot C + A \cdot M + M \cdot C + C \cdot A?$$

(A) 62 **(B)** 72 **(C)** 92 **(D)** 102 **(E)** 112

Strategy. The expression appears in the expansion of $(A + 1)(M + 1)(C + 1)$. This reduces the problem to maximizing a product of three numbers with a fixed sum.

Solution. Expanding gives

$$(A+1)(M+1)(C+1) = AMC + AM + MC + CA + A + M + C + 1.$$

Since $A + M + C = 12$, the last four terms add up to $A + M + C + 1 = 13$, so

$$AMC + AM + MC + CA = (A+1)(M+1)(C+1) - 13.$$

Let $p = A + 1$, $q = M + 1$, and $r = C + 1$. Then p, q, and r are positive integers with

$$p + q + r = (A + M + C) + 3 = 15.$$

By AM–GM,

$$\sqrt[3]{pqr} \leq \frac{p+q+r}{3} = 5, \qquad \text{so} \qquad pqr \leq 125,$$

with equality when $p = q = r = 5$, that is, $A = M = C = 4$. Therefore the maximum value is

$$125 - 13 = 112.$$

Answer: (E) 112. □

3. Sequences and Series

This section covers arithmetic and geometric sequences and sequences defined by recurrence relations.

3.1. **Arithmetic and Geometric Sequences.** An **arithmetic sequence** has a constant difference d between consecutive terms, its **common difference**. A **geometric sequence** has a constant ratio r between consecutive terms, its **common ratio**. The standard formulas are:

Sequence	**Definition**	**nth Term a_n**	**Sum of First n Terms S_n**
Arithmetic	Constant difference d	$a_n = a_1 + (n-1)d$	$S_n = \frac{n}{2}(a_1 + a_n) = \frac{n}{2}\big(2a_1 + (n-1)d\big)$
Geometric	Constant ratio r	$a_n = a_1 r^{n-1}$	$S_n = a_1 \frac{r^n - 1}{r - 1}$ $(r \neq 1)$

An infinite geometric series with first term $a_1 \neq 0$ has a sum exactly when $|r| < 1$, and then the sum is

$$S_\infty = \frac{a_1}{1-r}.$$

Of three consecutive terms of an arithmetic sequence, the middle term is the average of the other two. Writing the three terms as $x-d$, x, $x+d$ makes their sum $3x$.

Example.

Problem 3.1.1 (2006 AMC 10A, Problem #19)**.** How many non-similar triangles have angles whose degree measures are distinct positive integers in arithmetic progression?
(A) 0 **(B)** 1 **(C)** 59 **(D)** 89 **(E)** 178

Strategy. Write the angles as $x-d$, x, $x+d$ with d a positive integer. Then use the angle sum of a triangle and the condition that the angles are distinct positive integers.

Solution. List the angles in increasing order as $x-d$, x, $x+d$, where d is a positive integer because the angles are distinct integers. The angles of a triangle add to $180°$, so $3x = 180$ and $x = 60$. The angles are $60-d$, 60, and $60+d$. They are all positive exactly when $60-d > 0$, so $d \in \{1, 2, \ldots, 59\}$. Two triangles are similar exactly when they have the same three angles, and different values of d give different smallest angles $60-d$. So the 59 values of d give 59 non-similar triangles.

Answer: (C) 59. □

3.2. **Recurrence Relations.** A **recurrence relation** defines each term of a sequence from the terms before it. For example, the Fibonacci sequence satisfies $F_n = F_{n-1} + F_{n-2}$ with $F_1 = F_2 = 1$. Many recurrences can be handled by computing the first few terms and finding a pattern, or by computing terms until the required one is reached.

Some recurrences become simpler after a substitution. Taking reciprocals can turn a recurrence into the recurrence of an arithmetic sequence.

Example.

Problem 3.2.1. A sequence is defined by $a_1 = 1$ and

$$a_n = \frac{a_{n-1}}{1+3a_{n-1}} \quad (n \geq 2).$$

Find a_{10}.

Strategy. Take reciprocals of both sides. The fraction $\frac{1+3a_{n-1}}{a_{n-1}}$ splits into two simple terms.

Solution. If $a_{n-1} > 0$, then $a_n > 0$ as well. Since $a_1 = 1 > 0$, every term is positive, so the reciprocals are defined. Taking reciprocals gives

$$\frac{1}{a_n} = \frac{1+3a_{n-1}}{a_{n-1}} = \frac{1}{a_{n-1}} + 3.$$

Let $b_n = \frac{1}{a_n}$. Then $b_n = b_{n-1} + 3$ and $b_1 = 1$, so (b_n) is an arithmetic sequence with common difference 3. Hence $b_n = 1 + 3(n-1) = 3n-2$ and $a_n = \frac{1}{3n-2}$. In particular, $a_{10} = \frac{1}{28}$.

Answer: $\dfrac{1}{28}$. □

4. Divisibility and Primes

This section covers divisibility, prime factorization, the greatest common divisor and least common multiple, and the divisors of an integer.

4.1. **Primes and Factorization.** An integer n is **divisible** by a nonzero integer d if n/d is an integer. Then d **divides** n, written $d \mid n$, and d is a **divisor** (or **factor**) of n. A **prime** is a positive integer greater than 1 with exactly two positive divisors, 1 and itself. A positive integer greater than 1 that is not prime is **composite**. The number 1 is neither prime nor composite.

Theorem 4.1.1 (Fundamental Theorem of Arithmetic)**.** *Every integer $n > 1$ can be written as a product of primes, and this* ***prime factorization*** *is unique up to the order of the factors.*

4.2. **Divisibility Rules.**

Divisor	Test
2	Last digit is even $(0, 2, 4, 6, 8)$.
3	Sum of digits is divisible by 3.
4	Last two digits form a number divisible by 4.
5	Last digit is 0 or 5.
6	Divisible by both 2 and 3.
8	Last three digits form a number divisible by 8.
9	Sum of digits is divisible by 9.
10	Last digit is 0.
11	Alternating sum of digits is divisible by 11.

The tests for 3 and 9 come from the following fact. Write N in base 10 as

$$N = d_m 10^m + \cdots + d_1 \cdot 10 + d_0.$$

Each number $10^k - 1 = 99\ldots9$ is divisible by 9, so

$$N - (d_m + \cdots + d_1 + d_0) = d_m(10^m - 1) + \cdots + d_1(10 - 1)$$

is divisible by 9. Hence N and the sum of its digits leave the same remainder upon division by 9, and also upon division by 3. The tests for 4 and 8 hold because 100 is divisible by 4 and 1000 is divisible by 8.

Example.

Problem 4.2.1 (2010 AIME II, Problem #1)**.** Let N be the greatest integer multiple of 36 all of whose digits are even and no two of whose digits are the same. Find the remainder when N is divided by 1000.

Strategy. Since $36 = 4 \cdot 9$, the number N must pass the tests for 4 and for 9. The test for 9 decides which even digits N uses, and the test for 4 decides their order.

Solution. Since $36 = 2^2 \cdot 3^2$, the Fundamental Theorem of Arithmetic shows that N is a multiple of 36 exactly when both 4 and 9 divide N. The even digits are 0, 2, 4, 6, and 8, so N has at most five digits, and a number with more digits is larger. By the test for 9, the sum of the digits of N is a multiple of 9. The five even digits have sum 20, which is not a multiple of 9. Leaving out one digit e gives the sum $20 - e$, which is a multiple of 9 only for $e = 2$. So N has the four digits 0, 4, 6, and 8. The largest number with these digits is 8640, and its last two digits form 40, which is

divisible by 4. So 8640 also passes the test for 4, and $N = 8640 = 36 \cdot 240$. The remainder when N is divided by 1000 is 640.

Answer: 640. □

4.3. **GCD, LCM, and the Euclidean Algorithm.** For nonzero integers a and b:

- the **greatest common divisor** $\gcd(a, b)$ is the largest integer that divides both;
- the **least common multiple** $\text{lcm}(a, b)$ is the smallest positive integer that is a multiple of both.

Two integers are **relatively prime** if their greatest common divisor is 1.

For positive a and b, write $a = \prod p_i^{\alpha_i}$ and $b = \prod p_i^{\beta_i}$ over the same set of primes, allowing exponents equal to 0. Then

$$\gcd(a, b) = \prod p_i^{\min(\alpha_i, \beta_i)}, \qquad \text{lcm}(a, b) = \prod p_i^{\max(\alpha_i, \beta_i)}.$$

The **Euclidean Algorithm** finds $\gcd(a, b)$ without factoring. It uses the fact that $\gcd(a, b) = \gcd(b, r)$, where r is the remainder when a is divided by b. Repeat this step until the remainder is 0; the last nonzero remainder is the greatest common divisor. For example, for $\gcd(270, 192)$:

$$\begin{aligned} 270 &= 1 \cdot 192 + 78, \\ 192 &= 2 \cdot 78 + 36, \\ 78 &= 2 \cdot 36 + 6, \\ 36 &= 6 \cdot 6 + 0. \end{aligned}$$

The last nonzero remainder is 6, so $\gcd(270, 192) = 6$.

Working backward through these steps writes the greatest common divisor in the form $ax + by$ with integers x and y. This is the **Extended Euclidean Algorithm**. Here, substituting $36 = 192 - 2 \cdot 78$ and then $78 = 270 - 192$ into $6 = 78 - 2 \cdot 36$ gives

$$6 = 5 \cdot 78 - 2 \cdot 192 = 5 \cdot 270 - 7 \cdot 192.$$

Section 5.2 uses this to solve congruences.

Example.

Problem 4.3.1 (2006 AMC 10A, Problem #22)**.** Two farmers agree that pigs are worth \$300 and that goats are worth \$210. When one farmer owes the other money, he pays the debt in pigs or goats, with "change" received in the form of goats or pigs as necessary. (For example, a \$390 debt could be paid with two pigs, with one goat received in change.) What is the amount of the smallest positive debt that can be resolved in this way?

(A) \$5 **(B)** \$10 **(C)** \$30 **(D)** \$90 **(E)** \$210

Strategy. A debt that can be resolved has the form $300p+210g$ for integers p and g, where a negative number counts animals received in change. Every such amount is a multiple of $\gcd(300, 210)$, and working backward through the Euclidean Algorithm writes the greatest common divisor itself in this form.

Solution. Paying p pigs and g goats, where a negative number means animals received in change, resolves a debt of $300p+210g$ dollars. A common divisor of 300 and 210 divides every such amount. The Euclidean Algorithm gives

$$\begin{aligned} 300 &= 1 \cdot 210 + 90, \\ 210 &= 2 \cdot 90 + 30, \\ 90 &= 3 \cdot 30 + 0, \end{aligned}$$

so $\gcd(300, 210) = 30$, and every debt that can be resolved is a multiple of 30. Working backward,

$$30 = 210 - 2 \cdot 90 = 210 - 2\,(300 - 210) = 3 \cdot 210 - 2 \cdot 300.$$

So a debt of \$30 is resolved by paying three goats and receiving two pigs in change. No smaller positive amount is a multiple of 30, so the smallest positive debt is \$30.

Answer: (C) \$30. □

4.4. **Counting and Summing Divisors.** Let $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ be the prime factorization of n, where $p_1, p_2, \ldots, p_k$ are different primes. Write $\tau(n)$ for the number of positive divisors of n and $\sigma(n)$ for their sum.

Proposition 4.4.1 (Number of Divisors)**.**

$$\tau(n) = (a_1 + 1)(a_2 + 1) \cdots (a_k + 1).$$

Proof. By the Fundamental Theorem of Arithmetic, every positive divisor of n is $p_1^{b_1} \cdots p_k^{b_k}$ for exactly one choice of exponents with $0 \le b_i \le a_i$. There are $a_i + 1$ choices for each b_i. □

Proposition 4.4.2 (Sum of Divisors)**.**

$$\sigma(n) = \left(1 + p_1 + \cdots + p_1^{a_1}\right) \cdots \left(1 + p_k + \cdots + p_k^{a_k}\right) = \prod_{i=1}^{k} \frac{p_i^{a_i+1} - 1}{p_i - 1}.$$

Proof. Expanding the product gives one term $p_1^{b_1} \cdots p_k^{b_k}$ for each choice of exponents with $0 \le b_i \le a_i$, that is, each positive divisor of n exactly once. The second form follows from the formula for the sum of a geometric sequence. □

Examples.

Problem 4.4.3 (2021 Fall AMC 10B, Problem #6)**.** The least positive integer with exactly 2021 distinct positive divisors can be written in the form $m \cdot 6^k$, where m and k are integers and 6 is not a divisor of m. What is $m + k$?

(A) 47 **(B)** 58 **(C)** 59 **(D)** 88 **(E)** 90

Strategy. By the formula for the number of divisors, the exponents in the prime factorization, each increased by 1, multiply to $2021 = 43 \cdot 47$. Since 43 and 47 are prime, only two patterns of exponents are possible. For the smallest number, use the smallest primes and give the larger exponent to the smaller prime.

Solution. Suppose n has exactly 2021 positive divisors. By the formula for the number of divisors, the exponents in the prime factorization of n, each increased by 1, multiply to $2021 = 43 \cdot 47$, where 43 and 47 are prime. Each of these factors is at least 2, so either n is a prime to the power 2020, or $n = p^{46} q^{42}$ for distinct primes p and q.

In the first case, $n \ge 2^{2020}$. In the second case, replacing p or q by a smaller prime not already used makes n smaller, so the least such n uses the primes 2 and 3. Since $2^4 < 3^4$,

$$2^{46} \cdot 3^{42} = 2^{42} 3^{42} \cdot 2^4 < 2^{42} 3^{42} \cdot 3^4 = 2^{42} \cdot 3^{46},$$

so the larger exponent goes to 2. Also $2^{46} \cdot 3^{42} < 2^{46} \cdot 4^{42} = 2^{130} < 2^{2020}$. Therefore the least positive integer with exactly 2021 divisors is

$$2^{46} \cdot 3^{42} = 2^4 \cdot (2 \cdot 3)^{42} = 16 \cdot 6^{42}.$$

Since 16 is not divisible by 3, it is not divisible by 6, so $m = 16$ and $k = 42$, and $m + k = 58$.

Answer: (B) 58. □

Problem 4.4.4 (2021 AMC 10B, Problem #12)**.** Let $N = 34 \cdot 34 \cdot 63 \cdot 270$. What is the ratio of the sum of the odd divisors of N to the sum of the even divisors of N?

(A) 1 : 16 **(B)** 1 : 15 **(C)** 1 : 14 **(D)** 1 : 8 **(E)** 1 : 3

Strategy. Write $N = 2^3 M$ with M odd. The odd divisors of N are the divisors of M, and in the formula for the sum of the divisors the factor for the prime 2 separates from the rest.

Solution. Since $34 = 2 \cdot 17$, $63 = 3^2 \cdot 7$, and $270 = 2 \cdot 3^3 \cdot 5$,

$$N = 2^3 \cdot 3^5 \cdot 5 \cdot 7 \cdot 17^2 = 2^3 M, \qquad \text{where } M = 3^5 \cdot 5 \cdot 7 \cdot 17^2 \text{ is odd.}$$

The odd divisors of N are exactly the divisors of M, so their sum is $\sigma(M)$. By the formula for the sum of the divisors,

$$\sigma(N) = (1 + 2 + 4 + 8)\,\sigma(M) = \frac{2^4 - 1}{2 - 1}\,\sigma(M) = 15\,\sigma(M).$$

Every divisor of N is either odd or even, so the sum of the even divisors is $15\,\sigma(M) - \sigma(M) = 14\,\sigma(M)$. The ratio is $\sigma(M) : 14\,\sigma(M) = 1 : 14$.

Answer: (C) 1 : 14. □

5. Modular Arithmetic

This section covers congruences, which describe arithmetic with remainders, and Fermat's Little Theorem.

5.1. **Congruences.**

Definition 5.1.1. Let m be a positive integer. Two integers a and b are **congruent modulo** m, written $a \equiv b \pmod{m}$, if $m \mid (a - b)$. Equivalently, a and b leave the same remainder upon division by m.

Congruences can be added, subtracted, and multiplied. If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then

$$a + c \equiv b + d, \qquad a - c \equiv b - d, \qquad ac \equiv bd \pmod{m}.$$

Multiplying a congruence by itself repeatedly gives $a^k \equiv b^k \pmod{m}$ for every positive integer k.

There are only m possible remainders, so the remainders of $a, a^2, a^3, \ldots$ modulo m eventually repeat in a cycle. The units digit of a positive integer is its remainder modulo 10, and the number formed by its last two digits is its remainder modulo 100.

Example.

Problem 5.1.2 (2016 AMC 10B, Problem #8)**.** What is the tens digit of $2015^{2016} - 2017$?

(A) 0 **(B)** 1 **(C)** 3 **(D)** 5 **(E)** 8

Strategy. The tens digit is determined by the last two digits, that is, by the remainder modulo 100. Reduce 2015 modulo 100 and find the cycle of the powers of 15.

Solution. Work modulo 100. Since $2015 \equiv 15$, the powers of 2015 leave the same remainders as the powers of 15:

$$15^2 = 225 \equiv 25, \quad 15^3 \equiv 25 \cdot 15 = 375 \equiv 75, \quad 15^4 \equiv 75 \cdot 15 = 1125 \equiv 25 \pmod{100}.$$

From the second power on, the remainders alternate between 25 and 75, with 25 for even exponents: indeed $25^2 = 625 \equiv 25$, so $15^{2j} = (15^2)^j \equiv 25^j \equiv 25 \pmod{100}$ for every positive integer j. Since 2016 is even, $2015^{2016} \equiv 25 \pmod{100}$. Also $2017 \equiv 17$, so

$$2015^{2016} - 2017 \equiv 25 - 17 = 8 \pmod{100}.$$

The number $2015^{2016} - 2017$ is positive, so its last two digits are 0 and 8, and its tens digit is 0.

Answer: (A) 0. □

5.2. **Solving Linear Congruences.** A **linear congruence** has the form $ax \equiv b \pmod{m}$. An **inverse** of a modulo m is an integer a^{-1} with $a \cdot a^{-1} \equiv 1 \pmod{m}$.

- A solution exists if and only if $\gcd(a, m) \mid b$.
- If $\gcd(a, m) = 1$, then a has an inverse modulo m, unique modulo m, and the solutions are $x \equiv a^{-1}b \pmod{m}$.
- To find a^{-1}, use the Extended Euclidean Algorithm to find integers u and v with $au+mv = 1$. Then $au \equiv 1 \pmod{m}$, so u is an inverse of a.
- If $d = \gcd(a, m) > 1$ and $d \mid b$, divide a, b, and m by d. The congruence $\frac{a}{d}x \equiv \frac{b}{d} \pmod{\frac{m}{d}}$ has the same solutions, and $\gcd\left(\frac{a}{d}, \frac{m}{d}\right) = 1$.

Example.

Problem 5.2.1 (2003 AMC 10A, Problem #25)**.** Let n be a 5-digit number, and let q and r be the quotient and remainder, respectively, when n is divided by 100. For how many values of n is $q + r$ divisible by 11?

(A) 8180 **(B)** 8181 **(C)** 8182 **(D)** 9000 **(E)** 9090

Strategy. Write $n = 100q + r$, where q is formed by the first three digits and r by the last two digits. For each q, count the r with $r \equiv -q \pmod{11}$.

Solution. Every 5-digit n can be written as $n = 100q + r$ in exactly one way, with

$$q \in \{100, 101, \ldots, 999\} \quad (900 \text{ values}), \qquad r \in \{0, 1, \ldots, 99\} \quad (100 \text{ values}).$$

The condition is

$$q + r \equiv 0 \pmod{11}, \quad \text{that is,} \quad r \equiv -q \pmod{11}.$$

Fix q, and let k be the remainder of $-q$ modulo 11, so $k \in \{0, 1, \ldots, 10\}$. Count the r from 0 to 99 with $r \equiv k \pmod{11}$:

- if $k = 0$, then $r \in \{0, 11, 22, \ldots, 99\}$, which gives 10 values;
- if $k \in \{1, \ldots, 10\}$, then $r \in \{k, k + 11, \ldots, k + 88\}$, which gives 9 values.

So each q that is a multiple of 11 has 10 valid values of r, and every other q has 9.

The multiples of 11 from 100 to 999 run from $110 = 11 \cdot 10$ to $990 = 11 \cdot 90$, so there are $90 - 10 + 1 = 81$ of them. The number of valid pairs (q, r) is

$$81 \cdot 10 + (900 - 81) \cdot 9 = 810 + 7371 = 8181.$$

Each pair gives exactly one 5-digit n, so there are 8181 such numbers.

Answer: (B) 8181. □

5.3. **Fermat's Little Theorem.**

Theorem 5.3.1 (Fermat's Little Theorem)**.** *Let p be a prime. Then $a^p \equiv a \pmod{p}$ for every integer a, and $a^{p-1} \equiv 1 \pmod{p}$ if $p \nmid a$.*

Fermat's Little Theorem reduces large exponents modulo a prime: if $p \nmid a$, then $a^k \equiv a^{k'} \pmod{p}$ whenever k and k' leave the same remainder upon division by $p - 1$.

Example.

Problem 5.3.2 (1972 AHSME, Problem #31)**.** When the number 2^{1000} is divided by 13, the remainder in the division is:

(A) 1 **(B)** 2 **(C)** 3 **(D)** 7 **(E)** 11

Strategy. Since 13 is prime and does not divide 2, Fermat's Little Theorem gives $2^{12} \equiv 1 \pmod{13}$. Reduce the exponent modulo 12.

Solution. By Fermat's Little Theorem, $2^{12} \equiv 1 \pmod{13}$. Since $1000 = 12 \cdot 83 + 4$,

$$2^{1000} = (2^{12})^{83} \cdot 2^4 \equiv 1^{83} \cdot 16 \equiv 3 \pmod{13}.$$

Answer: (C) 3. □

6. Problems

Beginner.

1. (2021 AMC 10A, Problem #1) What is the value of

$$(2^2-2)-(3^2-3)+(4^2-4)?$$

(A) 1 **(B)** 2 **(C)** 5 **(D)** 8 **(E)** 12

2. (2021 AMC 10A, Problem #4) A cart rolls down a hill, traveling 5 inches the first second and accelerating so that during each successive 1-second time interval, it travels 7 inches more than during the previous 1-second interval. The cart takes 30 seconds to reach the bottom of the hill. How far, in inches, does it travel?

(A) 215 **(B)** 360 **(C)** 2992 **(D)** 3195 **(E)** 3242

3. Factor the polynomial $x^4-3x^3-4x^2+12x$.

4. (2010 AMC 10B, Problem #8) A ticket to a school play costs x dollars, where x is a whole number. A group of 9th graders buys tickets costing a total of \$48, and a group of 10th graders buys tickets costing a total of \$64. How many values for x are possible?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

5. (2009 AMC 10B, Problem #21) What is the remainder when

$$3^0+3^1+3^2+\cdots+3^{2009}$$

is divided by 8?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 4 **(E)** 6

Intermediate.

6. (2010 AMC 10A, Problem #21) The polynomial

$$x^3-ax^2+bx-2010$$

has three positive integer zeros. What is the smallest possible value of a?

(A) 78 **(B)** 88 **(C)** 98 **(D)** 108 **(E)** 118

7. (2016 AMC 10B, Problem #16) The sum of an infinite geometric series is a positive number S, and the second term in the series is 1. What is the smallest possible value of S?

(A) $\frac{1+\sqrt{5}}{2}$ **(B)** 2 **(C)** $\sqrt{5}$ **(D)** 3 **(E)** 4

8. (2019 AMC 10B, Problem #14) The base-ten representation for 19! is

$$121{,}6T5{,}100{,}40M{,}832{,}H00,$$

where T, M, and H denote digits that are not given. What is $T+M+H$?

(A) 3 **(B)** 8 **(C)** 12 **(D)** 14 **(E)** 17

9. (2022 AMC 10A, Problem #7) The least common multiple of a positive integer n and 18 is 180, and the greatest common divisor of n and 45 is 15. What is the sum of the digits of n?

(A) 3 **(B)** 6 **(C)** 8 **(D)** 9 **(E)** 12

10. (2020 AMC 10A, Problem #24) Let n be the least positive integer greater than 1000 for which $\gcd(63, n+120) = 21$ and $\gcd(n+63, 120) = 60$. What is the sum of the digits of n?

(A) 12 **(B)** 15 **(C)** 18 **(D)** 21 **(E)** 24

11. (2017 AMC 10B, Problem #23) Let

$$N = 123456789101112\ldots4344$$

be the 79-digit number that is formed by writing the integers from 1 to 44 in order, one after the other. What is the remainder when N is divided by 45?

(A) 1 **(B)** 4 **(C)** 9 **(D)** 18 **(E)** 44

12. For how many positive integers a is $a^4 - 3a^2 + 9$ a prime number?

Advanced.

13. (2019 AMC 10A, Problem #15) A sequence of numbers is defined recursively by $a_1 = 1$, $a_2 = \frac{3}{7}$, and

$$a_n = \frac{a_{n-2} \cdot a_{n-1}}{2a_{n-2} - a_{n-1}}$$

for all $n \geq 3$. Then a_{2019} can be written as $\frac{p}{q}$, where p and q are relatively prime positive integers. What is $p + q$?

(A) 2020 **(B)** 4039 **(C)** 6057 **(D)** 6061 **(E)** 8078

14. (2019 AMC 10A, Problem #24) Let p, q, and r be the distinct roots of the polynomial

$$x^3 - 22x^2 + 80x - 67.$$

There exist real numbers A, B, and C such that

$$\frac{1}{s^3 - 22s^2 + 80s - 67} = \frac{A}{s-p} + \frac{B}{s-q} + \frac{C}{s-r}$$

for all real numbers s with $s \notin \{p, q, r\}$. What is

$$\frac{1}{A} + \frac{1}{B} + \frac{1}{C}?$$

(A) 243 **(B)** 244 **(C)** 245 **(D)** 246 **(E)** 247

15. If $x^3 - 12x^2 + ax - 64$ has only real, nonnegative roots, find all possible values of a.

16. (2020 AMC 10B, Problem #22) What is the remainder when $2^{202} + 202$ is divided by $2^{101} + 2^{51} + 1$?

(A) 100 **(B)** 101 **(C)** 200 **(D)** 201 **(E)** 202

17. (1986 AIME, Problem #5) What is the largest positive integer n for which $n^3 + 100$ is divisible by $n + 10$?

18. Find the last two digits of 3^{3^3}.