POLYNOMIALS, INEQUALITIES, AND NUMBER THEORY FOR AMC 10: SOLUTIONS

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1. Solutions

Beginner.

1. (2021 AMC 10A, Problem #1) What is the value of

$$(2^2-2)-(3^2-3)+(4^2-4)?$$

(A) 1 **(B)** 2 **(C)** 5 **(D)** 8 **(E)** 12

Strategy. Evaluate the expression in each pair of parentheses first, then combine the three results from left to right.

Solution. The expressions in the three pairs of parentheses are $2^2-2=2$, $3^2-3=6$, and $4^2-4=12$. Therefore

$$(2^2-2)-(3^2-3)+(4^2-4)=2-6+12=8.$$

Answer: (D) 8. □

2. (2021 AMC 10A, Problem #4) A cart rolls down a hill, traveling 5 inches the first second and accelerating so that during each successive 1-second time interval, it travels 7 inches more than during the previous 1-second interval. The cart takes 30 seconds to reach the bottom of the hill. How far, in inches, does it travel?

(A) 215 **(B)** 360 **(C)** 2992 **(D)** 3195 **(E)** 3242

Strategy. The distances traveled in successive seconds form an arithmetic sequence. Add its 30 terms with the formula for the sum of an arithmetic sequence.

Solution. In the nth second the cart travels $a_n = 5+7(n-1)$ inches, so the distances form an arithmetic sequence with first term $a_1=5$ and common difference $d=7$. The last of the 30 terms is

$$a_{30}=5+29\cdot 7=208.$$

The total distance is the sum of the first 30 terms:

$$S_{30}=\frac{30}{2}(a_1+a_{30})=15\cdot(5+208)=15\cdot 213=3195.$$

Answer: (D) 3195. □

3. Factor the polynomial $x^4-3x^3-4x^2+12x$.

Strategy. Every term has a factor of x. Factor it out, then factor the remaining cubic by grouping.

Solution. Each term is divisible by x, so

$$x^4-3x^3-4x^2+12x=x(x^3-3x^2-4x+12).$$

Group the cubic into two pairs that share a factor:

$$x^3-3x^2-4x+12=x^2(x-3)-4(x-3)=(x-3)(x^2-4).$$

Finally, $x^2-4=(x-2)(x+2)$ is a difference of squares. Therefore

$$x^4-3x^3-4x^2+12x=x(x-3)(x-2)(x+2).$$

Answer: $x(x-3)(x-2)(x+2)$. □

4. (2010 AMC 10B, Problem #8) A ticket to a school play costs x dollars, where x is a whole number. A group of 9th graders buys tickets costing a total of \$48, and a group of 10th graders buys tickets costing a total of \$64. How many values for x are possible?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

Strategy. Each group buys a whole number of tickets, so x divides both 48 and 64. Compare the prime factorizations of 48 and 64.

Solution. The 9th graders buy $\frac{48}{x}$ tickets and the 10th graders buy $\frac{64}{x}$ tickets, so x is a positive divisor of both $48 = 2^4 \cdot 3$ and $64 = 2^6$. A common divisor cannot have the prime factor 3, since 3 does not divide 64, and its power of 2 is at most 2^4. So the possible prices are $x = 2^e$ with $0 \le e \le 4$, which are exactly the divisors of $\gcd(48, 64) = 2^4 = 16$. By the formula for the number of divisors, $16 = 2^4$ has $4 + 1 = 5$ divisors: 1, 2, 4, 8, and 16. Each of them works; for example, $x = 16$ means 3 tickets for the 9th graders and 4 for the 10th graders.

Answer: (E) 5. □

5. (2009 AMC 10B, Problem #21) What is the remainder when

$$3^0 + 3^1 + 3^2 + \cdots + 3^{2009}$$

is divided by 8?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 4 **(E)** 6

Strategy. Find the cycle of the powers of 3 modulo 8. Since $3^2 = 9 \equiv 1 \pmod 8$, the remainders alternate between 1 and 3, so the terms can be grouped in consecutive pairs whose sums all leave the same remainder.

Solution. Since $3^2 = 9 \equiv 1 \pmod 8$, for every integer $k \ge 0$,

$$3^{2k} = (3^2)^k \equiv 1 \qquad \text{and} \qquad 3^{2k+1} = 3 \cdot (3^2)^k \equiv 3 \pmod 8.$$

The sum has 2010 terms, from 3^0 to 3^{2009}, so it splits into the 1005 pairs $3^{2k} + 3^{2k+1}$ with $k = 0, 1, \ldots, 1004$. Each pair leaves the remainder $1 + 3 = 4$, so

$$3^0 + 3^1 + \cdots + 3^{2009} \equiv 1005 \cdot 4 = 4020 \pmod 8.$$

Since $4020 = 8 \cdot 502 + 4$, the remainder is 4.

Answer: (D) 4. □

Intermediate.

6. (2010 AMC 10A, Problem #21) The polynomial

$$x^3 - ax^2 + bx - 2010$$

has three positive integer zeros. What is the smallest possible value of a?

(A) 78 **(B)** 88 **(C)** 98 **(D)** 108 **(E)** 118

Strategy. By Vieta's formulas, a is the sum of the zeros and 2010 is their product. Factor 2010 and decide where its largest prime factor must go.

Solution. Let the zeros be the positive integers r, s, and t. By Vieta's formulas,

$$a = r + s + t \qquad \text{and} \qquad rst = 2010 = 2 \cdot 3 \cdot 5 \cdot 67.$$

Since 67 is prime, it divides one of the zeros, say t, so t is a multiple of 67. If $t \ge 2 \cdot 67 = 134$, then $a \ge 134 + 1 + 1 = 136$. If $t = 67$, then $rs = 30$. The pairs with product 30 are $1 \cdot 30$,

$2 \cdot 15$, $3 \cdot 10$, and $5 \cdot 6$, with sums 31, 17, 13, and 11. The smallest sum is $5+6=11$, which gives $a = 5+6+67 = 78$. This is less than 136, so it is the minimum, and it is attained:

$$(x-5)(x-6)(x-67) = x^3 - 78x^2 + 767x - 2010.$$

Answer: (A) 78. □

7. (2016 AMC 10B, Problem #16) The sum of an infinite geometric series is a positive number S, and the second term in the series is 1. What is the smallest possible value of S?

(A) $\dfrac{1+\sqrt{5}}{2}$ **(B)** 2 **(C)** $\sqrt{5}$ **(D)** 3 **(E)** 4

Strategy. Write S in terms of the common ratio r alone. The result has $r(1-r)$ in the denominator, and AM–GM bounds $r(1-r)$ because the sum $r + (1-r)$ is constant.

Solution. Let the first term be a and the common ratio be r. The series has a sum, so $|r| < 1$. The second term is $ar = 1$, so $r \neq 0$, $a = \frac{1}{r}$, and

$$S = \frac{a}{1-r} = \frac{1}{r(1-r)}.$$

Since $S > 0$ and $1 - r > 0$, the ratio r is positive, so $0 < r < 1$. Then r and $1-r$ are positive, and AM–GM gives

$$\sqrt{r(1-r)} \leq \frac{r+(1-r)}{2} = \frac{1}{2}, \qquad \text{so} \qquad r(1-r) \leq \frac{1}{4}.$$

Hence $S = \frac{1}{r(1-r)} \geq 4$. Equality holds when $r = 1-r$, that is, $r = \frac{1}{2}$. Then $a = 2$, and the series $2 + 1 + \frac{1}{2} + \frac{1}{4} + \cdots$ has sum 4.

Answer: (E) 4. □

8. (2019 AMC 10B, Problem #14) The base-ten representation for 19! is

$$121{,}6T5{,}100{,}40M{,}832{,}H00,$$

where T, M, and H denote digits that are not given. What is $T + M + H$?

(A) 3 **(B)** 8 **(C)** 12 **(D)** 14 **(E)** 17

Strategy. Count the factors of 5 in 19! to find H. Then 19! is divisible by 9 and by 11, so the tests for 9 and 11 give two conditions on T and M.

Solution. Among $1, 2, \ldots, 19$, the multiples of 5 are 5, 10, and 15, so the prime factorization of 19! contains 5^3, and it contains 2^3 as well. So 19! is divisible by $10^3 = 1000$, and its last three digits, $H00$, are 000. Hence $H = 0$.

Since 9 divides 19!, so does the sum of its digits,

$$\begin{aligned} &1+2+1+6+T+5+1+0+0+4+0+M+8+3+2+0+0+0 \\ &\quad = 33 + T + M. \end{aligned}$$

As $0 \leq T + M \leq 18$, this gives $T + M = 3$ or $T + M = 12$.

Since 11 divides 19!, so does the alternating sum of its digits,

$$\begin{aligned} &1-2+1-6+T-5+1-0+0-4+0-M+8-3+2-0+0-0 \\ &\quad = T - M - 7. \end{aligned}$$

As $-9 \leq T - M \leq 9$, this gives $T - M = 7$ or $T - M = -4$. The numbers $T+M$ and $T-M$ have the same parity, since their sum $2T$ is even. So either $T + M = 3$ and $T - M = 7$,

which gives $M = -2$, impossible for a digit, or $T + M = 12$ and $T - M = -4$, which gives $T = 4$ and $M = 8$. Therefore $T + M + H = 4 + 8 + 0 = 12$.

Answer: (C) 12. □

9. (2022 AMC 10A, Problem #7) The least common multiple of a positive integer n and 18 is 180, and the greatest common divisor of n and 45 is 15. What is the sum of the digits of n?

(A) 3 **(B)** 6 **(C)** 8 **(D)** 9 **(E)** 12

Strategy. Use the prime factorization of n. The least common multiple takes the larger exponent of each prime and the greatest common divisor takes the smaller one, so each condition fixes or bounds an exponent.

Solution. Since n divides $\operatorname{lcm}(n, 18) = 180 = 2^2 \cdot 3^2 \cdot 5$, the number n has the form $n = 2^a 3^b 5^c$. With $18 = 2 \cdot 3^2$, the condition on the least common multiple gives

$$\max(a, 1) = 2, \qquad \max(b, 2) = 2, \qquad \max(c, 0) = 1,$$

so $a = 2$, $b \leq 2$, and $c = 1$. With $45 = 3^2 \cdot 5$, the condition $\gcd(n, 45) = 15 = 3 \cdot 5$ gives $\min(b, 2) = 1$, so $b = 1$. Therefore $n = 2^2 \cdot 3 \cdot 5 = 60$, and the sum of its digits is $6 + 0 = 6$.

Answer: (B) 6. □

10. (2020 AMC 10A, Problem #24) Let n be the least positive integer greater than 1000 for which $\gcd(63, n + 120) = 21$ and $\gcd(n + 63, 120) = 60$. What is the sum of the digits of n?

(A) 12 **(B)** 15 **(C)** 18 **(D)** 21 **(E)** 24

Strategy. Split each condition into a divisibility condition and a non-divisibility condition. The divisibility conditions are congruences for n. Solve them, then test the candidates above 1000 against the non-divisibility conditions.

Solution. Since $63 = 3^2 \cdot 7$, the condition $\gcd(63, n+120) = 21$ says that 21 divides $n+120$ but 63 does not; equivalently, $21 \mid n + 120$ and $9 \nmid n + 120$. Since $120 = 2^3 \cdot 3 \cdot 5$, the condition $\gcd(n + 63, 120) = 60$ says that $60 \mid n + 63$ but $120 \nmid n + 63$.

The two divisibility conditions give

$$n \equiv -120 \equiv 6 \pmod{21} \qquad \text{and} \qquad n \equiv -63 \equiv 57 \pmod{60}.$$

Write $n = 60k + 57$. Modulo 21, $60 \equiv -3$ and $57 \equiv 15$, so the first congruence becomes $-3k + 15 \equiv 6$, that is, $3k \equiv 9 \pmod{21}$. Dividing the congruence and the modulus by 3 gives $k \equiv 3 \pmod{7}$. So $k = 7j + 3$ and

$$n = 60(7j + 3) + 57 = 420j + 237.$$

The values above 1000 are 1077, 1497, 1917, and so on. Check the two non-divisibility conditions in order.

- $n = 1077$: $n + 120 = 1197 = 63 \cdot 19$, so $\gcd(63, n + 120) = 63$. This fails.
- $n = 1497$: $n + 63 = 1560 = 120 \cdot 13$, so $\gcd(n + 63, 120) = 120$. This fails.
- $n = 1917$: $n + 120 = 2037 = 21 \cdot 97$, and 97 is not divisible by 3, so $9 \nmid 2037$ and $\gcd(63, 2037) = 21$. Also $n + 63 = 1980 = 60 \cdot 33$ with 33 odd, so $120 \nmid 1980$ and $\gcd(1980, 120) = 60$. Both conditions hold.

Therefore $n = 1917$, and the sum of its digits is $1 + 9 + 1 + 7 = 18$.

Answer: (C) 18. □

11. (2017 AMC 10B, Problem #23) Let

$$N = 123456789101112\ldots4344$$

be the 79-digit number that is formed by writing the integers from 1 to 44 in order, one after the other. What is the remainder when N is divided by 45?

(A) 1 **(B)** 4 **(C)** 9 **(D)** 18 **(E)** 44

Strategy. Since $45 = 5 \cdot 9$, find the remainders of N modulo 5 and modulo 9 separately: the last digit gives the first and the sum of the digits gives the second. Then combine them by solving a linear congruence.

Solution. Since 10 is divisible by 5, the number N leaves the same remainder upon division by 5 as its last digit, 4. So $N \equiv 4 \pmod 5$.

A number and the sum of its digits leave the same remainder upon division by 9. The digits of N are the digits of $1, 2, \ldots, 44$, and each of these numbers leaves the same remainder modulo 9 as the sum of its own digits. So, by the formula for the sum of an arithmetic sequence,

$$N \equiv 1 + 2 + \cdots + 44 = \frac{44}{2}(1 + 44) = 990 \equiv 0 \pmod 9,$$

since $990 = 9 \cdot 110$. Write $N = 9j$ for an integer j. Then $9j \equiv 4 \pmod 5$, and since $9 \equiv 4 \pmod 5$, this is $4j \equiv 4 \pmod 5$. Since $4 \cdot 4 - 3 \cdot 5 = 1$, the number 4 is an inverse of 4 modulo 5, and multiplying by it gives $j \equiv 16 \equiv 1 \pmod 5$. So $j = 5t + 1$ for an integer t, and $N = 45t + 9$. The remainder is 9.

Answer: (C) 9. □

12. For how many positive integers a is $a^4 - 3a^2 + 9$ a prime number?

Strategy. Complete the square to write the expression as a difference of squares, as in Sophie Germain's identity. A product of two positive integers is prime only if the smaller factor is 1.

Solution. Adding and subtracting $9a^2$ gives $a^4 - 3a^2 + 9 = (a^4 + 6a^2 + 9) - 9a^2$, so

$$a^4 - 3a^2 + 9 = (a^2 + 3)^2 - (3a)^2 = (a^2 - 3a + 3)(a^2 + 3a + 3).$$

For a positive integer a, both factors are positive integers, since $a^2 - 3a + 3 = \left(a - \frac{3}{2}\right)^2 + \frac{3}{4} > 0$, and the first factor is smaller than the second by $6a$. A product of two positive integers is prime only when the smaller one equals 1 and the larger one is prime. Now

$$a^2 - 3a + 3 = 1 \iff a^2 - 3a + 2 = 0 \iff (a - 1)(a - 2) = 0,$$

so $a = 1$ or $a = 2$. For $a = 1$ the value is $1 - 3 + 9 = 7$, and for $a = 2$ it is $16 - 12 + 9 = 13$. Both are prime, so there are 2 such integers.

Answer: 2. □

Advanced.

13. (2019 AMC 10A, Problem #15) A sequence of numbers is defined recursively by $a_1 = 1$, $a_2 = \frac{3}{7}$, and

$$a_n = \frac{a_{n-2} \cdot a_{n-1}}{2a_{n-2} - a_{n-1}}$$

for all $n \geq 3$. Then a_{2019} can be written as $\frac{p}{q}$, where p and q are relatively prime positive integers. What is $p + q$?

(A) 2020 **(B)** 4039 **(C)** 6057 **(D)** 6061 **(E)** 8078

Strategy. The recurrence is a product divided by a difference. Take reciprocals and look for a simpler pattern in $\frac{1}{a_n}$.

Solution. If a_{n-2} and a_{n-1} are nonzero, taking reciprocals of both sides of the recurrence gives

$$\frac{1}{a_n} = \frac{2a_{n-2} - a_{n-1}}{a_{n-2}\, a_{n-1}} = \frac{2}{a_{n-1}} - \frac{1}{a_{n-2}}.$$

Let $b_n = \frac{1}{a_n}$. Then $b_n = 2b_{n-1} - b_{n-2}$, which can be rewritten as

$$b_n - b_{n-1} = b_{n-1} - b_{n-2}.$$

So consecutive terms of (b_n) have a constant difference, and (b_n) is an arithmetic sequence. Its first term is $b_1 = 1$ and its common difference is $b_2 - b_1 = \frac{7}{3} - 1 = \frac{4}{3}$. This difference is positive, so every b_n is positive and every a_n is a positive number, which justifies the reciprocals. Therefore

$$b_{2019} = 1 + 2018 \cdot \frac{4}{3} = \frac{3 + 8072}{3} = \frac{8075}{3}, \qquad \text{so} \qquad a_{2019} = \frac{3}{8075}.$$

Since $8075 = 5^2 \cdot 17 \cdot 19$ is not divisible by 3, this fraction is in lowest terms, and $p + q = 3 + 8075 = 8078$.

Answer: (E) 8078. □

14. (2019 AMC 10A, Problem #24) Let p, q, and r be the distinct roots of the polynomial

$$x^3 - 22x^2 + 80x - 67.$$

There exist real numbers A, B, and C such that

$$\frac{1}{s^3 - 22s^2 + 80s - 67} = \frac{A}{s-p} + \frac{B}{s-q} + \frac{C}{s-r}$$

for all real numbers s with $s \notin \{p, q, r\}$. What is

$$\frac{1}{A} + \frac{1}{B} + \frac{1}{C}?$$

(A) 243 **(B)** 244 **(C)** 245 **(D)** 246 **(E)** 247

Strategy. Clear the denominators and substitute $s = p$: every term but one vanishes, which gives A in terms of the roots. Do the same for B and C, then use Vieta's formulas.

Solution. Since p, q, and r are the roots and the leading coefficient is 1,

$$s^3 - 22s^2 + 80s - 67 = (s-p)(s-q)(s-r).$$

Multiplying both sides of the given equation by this product gives

$$1 = A(s-q)(s-r) + B(s-p)(s-r) + C(s-p)(s-q)$$

for every s other than p, q, and r. By the Factor Theorem, each root of a nonzero polynomial gives it a linear factor, so a nonzero polynomial of degree at most 2 has at most 2 roots. The difference of the two sides has degree at most 2 and infinitely many roots, so it is zero for every s. In particular, the equation also holds at $s = p$, where the last two terms vanish:

$$1 = A(p-q)(p-r), \qquad \text{so} \qquad \frac{1}{A} = (p-q)(p-r).$$

In the same way, $\frac{1}{B} = (q-p)(q-r)$ and $\frac{1}{C} = (r-p)(r-q)$. Expanding the three products and adding,

$$\frac{1}{A} + \frac{1}{B} + \frac{1}{C} = p^2 + q^2 + r^2 - (pq + qr + rp).$$

By Vieta's formulas, $p + q + r = 22$ and $pq + qr + rp = 80$, so

$$p^2 + q^2 + r^2 = (p + q + r)^2 - 2(pq + qr + rp) = 484 - 160 = 324.$$

Therefore

$$\frac{1}{A} + \frac{1}{B} + \frac{1}{C} = 324 - 80 = 244.$$

Answer: (B) 244. □

15. If $x^3 - 12x^2 + ax - 64$ has only real, nonnegative roots, find all possible values of a.

Strategy. Vieta's formulas give the sum and the product of the roots. Compare them with AM–GM: they are exactly in the equality case.

Solution. Let the roots be $p, q, r \geq 0$. By Vieta's formulas,

$$p + q + r = 12, \qquad pqr = 64, \qquad pq + qr + rp = a.$$

By AM–GM,

$$4 = \frac{p + q + r}{3} \geq \sqrt[3]{pqr} = \sqrt[3]{64} = 4.$$

The two sides are equal, so equality holds in AM–GM, which happens only when $p = q = r$. Hence $p = q = r = 4$, and

$$a = pq + qr + rp = 3 \cdot 16 = 48.$$

Conversely, $a = 48$ works, since $x^3 - 12x^2 + 48x - 64 = (x-4)^3$ has the triple root 4.

Answer: 48. □

16. (2020 AMC 10B, Problem #22) What is the remainder when $2^{202} + 202$ is divided by $2^{101} + 2^{51} + 1$?

(A) 100 **(B)** 101 **(C)** 200 **(D)** 201 **(E)** 202

Strategy. Write $2^{202} + 1$ in the form $a^4 + 4b^4$ and use Sophie Germain's identity. One of the two factors is the divisor.

Solution. Take $a = 1$ and $b = 2^{50}$. Then $a^4 + 4b^4 = 1 + 4 \cdot 2^{200} = 2^{202} + 1$. Since $a^2 + 2b^2 = 1 + 2^{101}$ and $2ab = 2^{51}$, Sophie Germain's identity gives

$$2^{202} + 1 = (a^2 + 2b^2 - 2ab)(a^2 + 2b^2 + 2ab) = (2^{101} - 2^{51} + 1)(2^{101} + 2^{51} + 1).$$

So $2^{101} + 2^{51} + 1$ divides $2^{202} + 1$. Now

$$2^{202} + 202 = (2^{202} + 1) + 201,$$

and $0 \leq 201 < 2^{101} + 2^{51} + 1$, so the remainder is 201.

Answer: (D) 201. □

17. (1986 AIME, Problem #5) What is the largest positive integer n for which $n^3 + 100$ is divisible by $n + 10$?

Strategy. Use the sum of cubes to write $n^3 + 100$ as a multiple of $n + 10$ plus a constant. Then $n + 10$ must divide that constant.

Solution. By the sum of cubes identity, $n^3 + 1000 = (n + 10)(n^2 - 10n + 100)$, so

$$n^3 + 100 = (n + 10)(n^2 - 10n + 100) - 900.$$

The first term is a multiple of $n+10$, so $n+10$ divides n^3+100 exactly when $n+10$ divides 900. The largest divisor of 900 is 900 itself, so the largest n is $900 - 10 = 890$.

Answer: 890. □

18. Find the last two digits of 3^{3^3}.

Strategy. The last two digits are the remainder modulo 100. The exponent is $3^3 = 27$, so look for a power of 3 that is 1 modulo 100.

Solution. The tower means $3^{(3^3)} = 3^{27}$, and its last two digits are its remainder modulo 100. Building up powers,

$$3^5 = 243 \equiv 43, \qquad 3^{10} \equiv 43^2 = 1849 \equiv 49, \qquad 3^{20} \equiv 49^2 = 2401 \equiv 1 \pmod{100}.$$

Hence $3^{27} = 3^{20} \cdot 3^7 \equiv 3^7 = 2187 \equiv 87 \pmod{100}$, so the last two digits are 87.

Answer: 87. □

2. Answer Key

Problem	Letter	Value
1	D	8
2	D	3195
3		$x(x-3)(x-2)(x+2)$
4	E	5
5	D	4
6	A	78
7	E	4
8	C	12
9	B	6
10	C	18
11	C	9
12		2
13	E	8078
14	B	244
15		48
16	D	201
17		890
18		87