POLYNOMIALS, INEQUALITIES, AND NUMBER THEORY FOR AMC 10: PRACTICE SET SOLUTIONS

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1. Solutions

Beginner.

1. (2022 AMC 10B, Problem #4) A donkey suffers an attack of hiccups and the first hiccup happens at 4:00 one afternoon. Suppose that the donkey hiccups regularly every 5 seconds. At what time does the donkey's 700th hiccup occur?

(A) 15 seconds after 4:58 **(B)** 20 seconds after 4:58 **(C)** 25 seconds after 4:58
(D) 30 seconds after 4:58 **(E)** 35 seconds after 4:58

Strategy. The 700th hiccup comes 699 intervals after the first one, not 700.

Solution. The times of the hiccups form an arithmetic sequence with common difference $d = 5$ seconds, and in the formula $a_n = a_1 + (n-1)d$ the factor $n-1$ counts the intervals from the first term to the nth. Between the 1st and the 700th hiccup there are 699 intervals of 5 seconds, a total of $699 \cdot 5 = 3495$ seconds. Since $58 \cdot 60 = 3480$, this is 58 minutes and 15 seconds. So the 700th hiccup occurs 15 seconds after 4:58.

Answer: (A) 15 seconds after 4:58. □

2. (2018 AMC 10A, Problem #7) For how many (not necessarily positive) integer values of n is the following value an integer?

$$4000 \cdot \left(\frac{2}{5}\right)^n$$

(A) 3 **(B)** 4 **(C)** 6 **(D)** 8 **(E)** 9

Strategy. Write everything in prime factorizations. The expression is an integer exactly when no prime ends up with a negative exponent.

Solution. Since $4000 = 2^5 \cdot 5^3$,

$$4000 \cdot \left(\frac{2}{5}\right)^n = 2^{5+n} \cdot 5^{3-n}.$$

If $n > 3$, this is a power of 2 divided by a positive power of 5, which is not an integer because no power of 2 is divisible by 5. If $n < -5$, it is a power of 5 divided by a positive power of 2, which is not an integer for the same reason. If $-5 \le n \le 3$, both exponents are nonnegative and the value is an integer. That gives the 9 values $n = -5, -4, \ldots, 3$.

Answer: (E) 9. □

3. (2021 AMC 10A, Problem #10) Which of the following is equivalent to

$$(2+3)(2^2+3^2)(2^4+3^4)(2^8+3^8)(2^{16}+3^{16})(2^{32}+3^{32})(2^{64}+3^{64})?$$

(A) $3^{127}+2^{127}$ **(B)** $3^{127}+2^{127}+2\cdot 3^{63}+3\cdot 2^{63}$ **(C)** $3^{128}-2^{128}$ **(D)** $3^{128}+2^{128}$
(E) 5^{127}

Strategy. Multiplying the product by $3-2=1$ does not change it. The new factor $3-2$ combines with $3+2$ by the difference of squares, and each result combines with the next factor in the same way.

Solution. Since $3-2=1$, the product equals

$$(3-2)(3+2)(3^2+2^2)(3^4+2^4)(3^8+2^8)(3^{16}+2^{16})(3^{32}+2^{32})(3^{64}+2^{64}).$$

For every positive integer k, the difference of squares with $a = 3^k$ and $b = 2^k$ gives

$$(3^k-2^k)(3^k+2^k) = 3^{2k}-2^{2k}.$$

With $k = 1$, the first two factors multiply to $3^2 - 2^2$. With $k = 2$, multiplying by the next factor gives $3^4 - 2^4$. Each further factor doubles the exponent in the same way, for $k = 4$, 8, 16, 32, and finally 64:

$$(3^{64} - 2^{64})(3^{64} + 2^{64}) = 3^{128} - 2^{128}.$$

So the product equals $3^{128} - 2^{128}$.

Answer: (C) $3^{128} - 2^{128}$. □

4. (2013 AMC 10B, Problem #9) Three positive integers are each greater than 1, have a product of 27000, and are pairwise relatively prime. What is their sum?

(A) 100 **(B)** 137 **(C)** 156 **(D)** 160 **(E)** 165

Strategy. Factor 27000. Numbers that are pairwise relatively prime share no prime factor, so each prime power must go entirely to one of the three numbers.

Solution. Factor: $27000 = 27 \cdot 1000 = 2^3 \cdot 3^3 \cdot 5^3$. If two of the numbers were divisible by the same prime, they would not be relatively prime. So all of 2^3 goes into one number, all of 3^3 into one number, and all of 5^3 into one number. Each of the three numbers is greater than 1, so each receives at least one of these three prime powers, and therefore exactly one. The numbers are 8, 27, and 125, and their sum is $8 + 27 + 125 = 160$.

Answer: (D) 160. □

5. (1983 AHSME, Problem #14) The units digit of $3^{1001}7^{1002}13^{1003}$ is

(A) 1 **(B)** 3 **(C)** 5 **(D)** 7 **(E)** 9

Strategy. The units digit is the remainder modulo 10. The remainders of the powers of 3 and of 7 modulo 10 repeat in cycles of length 4, and $13 \equiv 3 \pmod{10}$.

Solution. The units digit of a positive integer is its remainder modulo 10, and congruences can be multiplied, so it is enough to find the remainder of each factor. Since $3^4 = 81 \equiv 1 \pmod{10}$, writing $k = 4q + r$ gives $3^k = (3^4)^q \cdot 3^r \equiv 3^r \pmod{10}$, so the remainders of the powers of 3 repeat in the cycle 3, 9, 7, 1. In the same way, $7^4 = 2401 \equiv 1 \pmod{10}$, and the remainders of the powers of 7 repeat in the cycle 7, 9, 3, 1. Since $1001 = 4 \cdot 250 + 1$ and $1002 = 4 \cdot 250 + 2$,

$$3^{1001} \equiv 3^1 = 3 \qquad \text{and} \qquad 7^{1002} \equiv 7^2 = 49 \equiv 9 \pmod{10}.$$

Also $13 \equiv 3 \pmod{10}$, so $13^{1003} \equiv 3^{1003}$, and since $1003 = 4 \cdot 250 + 3$, this is $3^3 = 27 \equiv 7$. Therefore

$$3^{1001}7^{1002}13^{1003} \equiv 3 \cdot 9 \cdot 7 = 189 \equiv 9 \pmod{10},$$

and the units digit is 9.

Answer: (E) 9. □

Intermediate.

6. (2017 AMC 10B, Problem #14) An integer N is selected at random in the range $1 \leq N \leq 2020$. What is the probability that the remainder when N^{16} is divided by 5 is 1?

(A) $\frac{1}{5}$ **(B)** $\frac{2}{5}$ **(C)** $\frac{3}{5}$ **(D)** $\frac{4}{5}$ **(E)** 1

Strategy. Since 5 is prime, Fermat's Little Theorem gives $N^4 \equiv 1 \pmod{5}$ whenever $5 \nmid N$. So the remainder of N^{16} depends only on whether 5 divides N.

Solution. If $5 \nmid N$, then Fermat's Little Theorem with the prime 5 gives $N^4 \equiv 1 \pmod 5$, so

$$N^{16} = \left(N^4\right)^4 \equiv 1^4 = 1 \pmod 5,$$

and the remainder is 1. If $5 \mid N$, then $5 \mid N^{16}$ and the remainder is 0. So the remainder is 1 exactly when N is not a multiple of 5. Of the 2020 integers from 1 to 2020, exactly $2020/5 = 404$ are multiples of 5, so $2020 - 404 = 1616$ are not. All 2020 values of N are equally likely, so the probability is

$$\frac{1616}{2020} = \frac{4}{5}.$$

Answer: (D) $\frac{4}{5}$. □

7. (2003 AMC 10B, Problem #25) How many distinct four-digit numbers are divisible by 3 and have 23 as their last two digits?

(A) 27 **(B)** 30 **(C)** 33 **(D)** 81 **(E)** 90

Strategy. Write the number as $100k + 23$, where k is the two-digit number formed by the first two digits, and reduce modulo 3.

Solution. A four-digit number ending in 23 has the form $N = 100k + 23$ with $10 \le k \le 99$. Since $100 \equiv 1$ and $23 \equiv 2 \pmod 3$,

$$N \equiv k + 2 \pmod 3.$$

So N is divisible by 3 exactly when $k \equiv 1 \pmod 3$. The test for 3 gives the same condition: if the digits of k are a and b, the digits of N add up to $a + b + 5$, and $a + b \equiv k \pmod 3$. The values of k are $10, 13, 16, \ldots, 97$, an arithmetic sequence with common difference 3 and

$$\frac{97 - 10}{3} + 1 = 30$$

terms.

Answer: (B) 30. □

8. (2009 AMC 10B, Problem #14) On Monday, Millie puts a quart of seeds, 25% of which are millet, into a bird feeder. On each successive day she adds another quart of the same mix of seeds without removing any seeds that are left. Each day the birds eat only 25% of the millet in the feeder, but they eat all of the other seeds. On which day, just after Millie has placed the seeds, will the birds find that more than half the seeds in the feeder are millet?

(A) Tuesday **(B)** Wednesday **(C)** Thursday **(D)** Friday **(E)** Saturday

Strategy. Let M_n be the amount of millet just after the seeds are placed on day n, and write a recurrence relation for it. The substitution $D_n = 1 - M_n$ turns the recurrence into that of a geometric sequence.

Solution. Number the days so that Monday is day 1, and let M_n be the amount of millet, in quarts, in the feeder just after Millie places the seeds on day n. The birds eat all of the other seeds every day, so at that moment the feeder holds exactly the $\frac{3}{4}$ quart of other seeds added that day. During day n the birds eat 25% of the millet and leave $\frac{3}{4}M_n$ quarts, and on day $n + 1$ Millie adds $\frac{1}{4}$ quart more. So

$$M_1 = \frac{1}{4} \qquad \text{and} \qquad M_{n+1} = \frac{3}{4}M_n + \frac{1}{4} \quad (n \ge 1).$$

More than half the seeds are millet exactly when there is more millet than other seeds, that is, when $M_n > \frac{3}{4}$.

Let $D_n = 1 - M_n$. Then

$$D_{n+1} = 1 - \frac{3}{4}M_n - \frac{1}{4} = \frac{3}{4}(1 - M_n) = \frac{3}{4}D_n,$$

so (D_n) is a geometric sequence with first term $D_1 = \frac{3}{4}$ and common ratio $\frac{3}{4}$. Hence

$$D_n = \left(\frac{3}{4}\right)^n \qquad \text{and} \qquad M_n = 1 - \left(\frac{3}{4}\right)^n.$$

The condition $M_n > \frac{3}{4}$ becomes $\left(\frac{3}{4}\right)^n < \frac{1}{4}$, that is, $4 \cdot 3^n < 4^n$. For $n = 4$ this fails, since $4 \cdot 81 = 324 > 256$, and for $n = 5$ it holds, since $4 \cdot 243 = 972 < 1024$. The numbers $\left(\frac{3}{4}\right)^n$ decrease as n grows, so the condition fails on days 1 to 4 and first holds on day 5, which is Friday.

Answer: (D) Friday. □

9. (2003 AMC 12B, Problem #18) Let x and y be positive integers such that $7x^5 = 11y^{13}$. The minimum possible value of x has a prime factorization $a^c b^d$. What is $a + b + c + d$?

(A) 30 **(B)** 31 **(C)** 32 **(D)** 33 **(E)** 34

Strategy. Compare the exponent of each prime on the two sides of $7x^5 = 11y^{13}$. The exponents of 7 and 11 in x must satisfy linear congruences modulo 13, which the inverse of 5 modulo 13 solves.

Solution. By the Fundamental Theorem of Arithmetic, the two sides of $7x^5 = 11y^{13}$ have the same exponent of every prime. Let p be a prime other than 7 and 11, with exponent e in x and f in y. Then $5e = 13f$, so dividing x by p^e and y by p^f divides both sides by the same number $p^{5e} = p^{13f}$. The equation still holds, and if p divides x, then $e \geq 1$ and x becomes smaller. So the minimum x has no prime factors other than 7 and 11. Write $x = 7^s 11^t$ and $y = 7^u 11^v$ with nonnegative integers s, t, u, v. Comparing the exponents of 7 and of 11 gives

$$5s + 1 = 13u \qquad \text{and} \qquad 5t = 13v + 1,$$

that is,

$$5s \equiv -1 \pmod{13} \qquad \text{and} \qquad 5t \equiv 1 \pmod{13}.$$

To find an inverse of 5 modulo 13, use the Extended Euclidean Algorithm: run the Euclidean Algorithm,

$$13 = 2 \cdot 5 + 3, \qquad 5 = 1 \cdot 3 + 2, \qquad 3 = 1 \cdot 2 + 1,$$

and work backward:

$$1 = 3 - 2 = 3 - (5 - 3) = 2 \cdot 3 - 5 = 2(13 - 2 \cdot 5) - 5 = 2 \cdot 13 - 5 \cdot 5.$$

So $5 \cdot (-5) \equiv 1 \pmod{13}$, and $-5 \equiv 8$ is an inverse of 5. Multiplying both congruences by 8 gives $s \equiv -8 \equiv 5$ and $t \equiv 8 \pmod{13}$. The least nonnegative solutions $s = 5$ and $t = 8$ work, since $5 \cdot 5 + 1 = 26 = 13 \cdot 2$ and $5 \cdot 8 = 40 = 13 \cdot 3 + 1$ give $u = 2$ and $v = 3$, and larger exponents make x larger. So the minimum is

$$x = 7^5 \cdot 11^8, \qquad \text{with} \qquad y = 7^2 \cdot 11^3.$$

Its prime factorization uses the primes 7 and 11 with the exponents 5 and 8, so

$$a + b + c + d = 7 + 11 + 5 + 8 = 31.$$

Answer: (B) 31. □

10. (1958 AHSME, Problem #46) For values of x less than 1 but greater than -4, the expression

$$\frac{x^2 - 2x + 2}{2x - 2}$$

has:

(A) no maximum or minimum value **(B)** a minimum value of 1
(C) a maximum value of 1 **(D)** a minimum value of -1
(E) a maximum value of -1

Strategy. Substitute $t = 1 - x$, which is positive on the given interval. The expression becomes the negative of a sum of two positive terms with a constant product, and that sum has a lower bound by AM–GM.

Solution. Let $t = 1 - x$. As x runs over the numbers with $-4 < x < 1$, t runs over the numbers with $0 < t < 5$. Completing the square gives $x^2 - 2x + 2 = (x-1)^2 + 1 = t^2 + 1$, and $2x - 2 = -2t$, so

$$\frac{x^2 - 2x + 2}{2x - 2} = \frac{t^2 + 1}{-2t} = -\left(\frac{t}{2} + \frac{1}{2t}\right).$$

The numbers $\frac{t}{2}$ and $\frac{1}{2t}$ are positive and their product is $\frac{1}{4}$. By AM–GM,

$$\frac{t}{2} + \frac{1}{2t} \;\geq\; 2\sqrt{\frac{t}{2} \cdot \frac{1}{2t}} = 1,$$

with equality exactly when $\frac{t}{2} = \frac{1}{2t}$, that is, $t = 1$. So the expression is at most -1, and it equals -1 at $t = 1$, that is, at $x = 0$, which lies in the interval. Hence -1 is the maximum value.

There is no minimum value. The expression is less than $-\frac{1}{2t}$, and as x approaches 1, t approaches 0 and $-\frac{1}{2t}$ becomes less than any given number.

Answer: (E) a maximum value of -1. □

Advanced.

11. (2016 AMC 10B, Problem #17) All the numbers 2, 3, 4, 5, 6, 7 are assigned to the six faces of a cube, one number to each face. For each of the eight vertices of the cube, a product of three numbers is computed, where the three numbers are the numbers assigned to the three faces that include that vertex. What is the greatest possible value of the sum of these eight products?

(A) 312 **(B)** 343 **(C)** 625 **(D)** 729 **(E)** 1680

Strategy. Group the faces into three pairs of opposite faces. Each vertex lies on exactly one face of each pair, so the sum of the eight products factors into a product of three sums whose total is fixed.

Solution. Let the numbers on the three pairs of opposite faces be a and b, c and d, and e and f. The three faces at a vertex are pairwise adjacent, so no two of them are opposite, and they contain exactly one face from each pair. Conversely, each of the $2 \cdot 2 \cdot 2 = 8$ ways to choose one face from each pair gives three faces that meet at exactly one vertex. Expanding a product of three sums gives one term for each way to choose one term from each sum, so the sum of the eight products is

$$ace + acf + ade + adf + bce + bcf + bde + bdf = (a+b)(c+d)(e+f).$$

The three factors are positive, and their sum is $2+3+4+5+6+7=27$. By AM–GM,

$$(a+b)(c+d)(e+f) \le \left(\frac{(a+b)+(c+d)+(e+f)}{3}\right)^3 = 9^3 = 729,$$

with equality when all three sums equal 9. The pairs $\{2,7\}$, $\{3,6\}$, and $\{4,5\}$ each have sum 9, so placing each of these pairs on opposite faces gives the sum 729. The greatest possible value is 729.

Answer: (D) 729. □

12. (2024 AMC 12B, Problem #17) Integers a and b are randomly chosen without replacement from the set of integers with absolute value not exceeding 10. What is the probability that the polynomial x^3+ax^2+bx+6 has 3 distinct integer roots?

(A) $\frac{1}{240}$ **(B)** $\frac{1}{221}$ **(C)** $\frac{1}{105}$ **(D)** $\frac{1}{84}$ **(E)** $\frac{1}{63}$

Strategy. If the roots are distinct integers r, s, t, Vieta's formulas give $r+s+t=-a$, $rs+st+tr=b$, and $rst=-6$. List the sets of three distinct integers with product -6 and keep those that give allowed values of a and b.

Solution. The integers with absolute value at most 10 are $-10,-9,\ldots,10$, which is 21 integers. Choosing a and then b without replacement gives $21\cdot 20 = 420$ equally likely ordered pairs (a,b) with $a \neq b$.

Suppose the polynomial has three distinct integer roots r, s, t. Its leading coefficient is 1, so $x^3+ax^2+bx+6=(x-r)(x-s)(x-t)$, and Vieta's formulas give

$$r+s+t=-a, \qquad rs+st+tr=b, \qquad rst=-6.$$

The absolute values of r, s, t multiply to 6, so they are 1, 2, 3 or 1, 1, 6. In the first case the product is -6 exactly when one or all three of the roots are negative, which gives four sets. In the second case the two roots with absolute value 1 are different, so they are 1 and -1, and then the third root is 6. The five sets give these values:

Roots	$a=-(r+s+t)$	$b=rs+st+tr$
1, 2, -3	0	-7
1, -2, 3	-2	-5
-1, 2, 3	-4	1
-1, -2, -3	6	11
-1, 1, 6	-6	-1

The value $b=11$ is out of range. The other four pairs satisfy $|a|\le 10$, $|b|\le 10$, and $a\neq b$, and for each of them the polynomial is $(x-r)(x-s)(x-t)$, which has three distinct integer roots. So 4 of the 420 pairs work, and the probability is

$$\frac{4}{420} = \frac{1}{105}.$$

Answer: (C) $\frac{1}{105}$. □

13. (2015 AIME I, Problem #3) There is a prime number p such that $16p+1$ is the cube of a positive integer. Find p.

Strategy. Write $16p=n^3-1$ and factor the right side by the difference of cubes. The factor n^2+n+1 is odd, so 16 divides $n-1$, and since p is prime the rest is forced.

Solution. Let $16p+1=n^3$, where n is a positive integer. By the difference of cubes,

$$16p = n^3 - 1 = (n-1)(n^2+n+1).$$

The factor $n^2+n+1 = n(n+1)+1$ is odd, since $n(n+1)$ is a product of two consecutive integers and hence even. By the Fundamental Theorem of Arithmetic, the power of 2 in $(n-1)(n^2+n+1)$ is then the power of 2 in $n-1$, and it is at least 2^4, so $16 \mid n-1$. Since $n^3 = 16p+1 > 1$, the number $n-1$ is positive, so $n-1 = 16k$ for a positive integer k. Then $16p = 16k(n^2+n+1)$, so

$$p = k(n^2+n+1).$$

Since p is prime and $n^2+n+1 > 1$, this forces $k=1$. Hence $n=17$ and

$$p = 17^2 + 17 + 1 = 307.$$

This number is prime: $307 < 19^2$, and 307 is not divisible by any of the primes 2, 3, 5, 7, 11, 13, 17. Indeed $16 \cdot 307 + 1 = 4913 = 17^3$.

Answer: 307. □

14. (2016 AMC 10A, Problem #22) For some positive integer n, the number $110n^3$ has 110 positive integer divisors, including 1 and the number $110n^3$. How many positive integer divisors does the number $81n^4$ have?

(A) 110 **(B)** 191 **(C)** 261 **(D)** 325 **(E)** 425

Strategy. By the formula for the number of divisors, the exponents in the prime factorization of $110n^3$, each increased by 1, multiply to $110 = 2 \cdot 5 \cdot 11$. Since 2, 5, and 11 divide $110n^3$, they already give three factors of at least 2, which leaves no room for another prime in n.

Solution. Write $n = 2^a 5^b 11^c m$, where a, b, $c \geq 0$ and m is a positive integer not divisible by 2, 5, or 11. Since $110 = 2 \cdot 5 \cdot 11$,

$$110n^3 = 2^{3a+1}\, 5^{3b+1}\, 11^{3c+1}\, m^3,$$

and a prime q with exponent $e_q \geq 1$ in m has exponent $3e_q$ in $110n^3$. By the formula for the number of divisors,

$$\tau(110n^3) = (3a+2)(3b+2)(3c+2) \prod_{q \mid m} (3e_q+1) = 110 = 2 \cdot 5 \cdot 11.$$

Every factor on the left is at least 2, so it has at least one prime factor, and by the Fundamental Theorem of Arithmetic the prime factors of all the factors together are 2, 5, and 11, three primes in all. The three factors $3a+2$, $3b+2$, $3c+2$ use up all three, so there is no factor $3e_q+1$, which means $m=1$, and $3a+2$, $3b+2$, $3c+2$ are 2, 5, 11 in some order. So a, b, c are 0, 1, 3 in some order.

Then $81n^4 = 3^4 \cdot 2^{4a} 5^{4b} 11^{4c}$, and 3 does not divide n. The numbers $4a+1$, $4b+1$, $4c+1$ are 1, 5, 13 in some order, so by the formula for the number of divisors again,

$$\tau(81n^4) = (4+1)(4a+1)(4b+1)(4c+1) = 5 \cdot 1 \cdot 5 \cdot 13 = 325.$$

Answer: (D) 325. □

15. (1977 AHSME, Problem #28) Let $g(x) = x^5 + x^4 + x^3 + x^2 + x + 1$. What is the remainder when the polynomial $g(x^{12})$ is divided by the polynomial $g(x)$?

(A) 6 **(B)** $5 - x$ **(C)** $4 - x + x^2$ **(D)** $3 - x + x^2 - x^3$
(E) $2 - x + x^2 - x^3 + x^4$

Strategy. Multiplying $g(x)$ by $x-1$ gives x^6-1. Write $g(x^{12})-6$ as a sum of terms $x^{12k}-1$, each a multiple of x^6-1 and so of $g(x)$, and use the uniqueness of the remainder.

Solution. Multiplying out, all the middle terms cancel:

$$(x-1)g(x) = (x-1)(x^5+x^4+x^3+x^2+x+1) = x^6-1.$$

Next,

$$g(x^{12}) = x^{60}+x^{48}+x^{36}+x^{24}+x^{12}+1,$$

so

$$g(x^{12})-6 = (x^{60}-1)+(x^{48}-1)+(x^{36}-1)+(x^{24}-1)+(x^{12}-1).$$

For every positive integer m, multiplying out shows that

$$y^m-1 = (y-1)(y^{m-1}+y^{m-2}+\cdots+y+1),$$

which is the formula for the sum of a geometric sequence written as a factorization. With $y=x^6$ and $m=2k$, each term $x^{12k}-1=(x^6)^{2k}-1$ is a multiple of $x^6-1=(x-1)g(x)$, and so a multiple of $g(x)$. Hence $g(x^{12})-6=g(x)Q(x)$ for some polynomial $Q(x)$, that is,

$$g(x^{12}) = g(x)Q(x)+6.$$

The polynomial 6 has degree 0, which is less than 5, the degree of $g(x)$. By the uniqueness in the Division Algorithm, the remainder is 6.

Answer: (A) 6. □

2. Answer Key

Problem	Letter	Value
1	A	15 seconds after 4:58
2	E	9
3	C	$3^{128} - 2^{128}$
4	D	160
5	E	9
6	D	$\frac{4}{5}$
7	B	30
8	D	Friday
9	B	31
10	E	a maximum value of -1
11	D	729
12	C	$\frac{1}{105}$
13		307
14	D	325
15	A	6