QUADRATICS, THE DISCRIMINANT, AND VIETA'S FORMULAS FOR AMC 10

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1. Solving Quadratic Equations

A quadratic equation can be solved by factoring, by completing the square, or by the quadratic formula. This section covers all three.

1.1. **Factoring.**

Definition 1.1.1. A **quadratic** in x is an expression ax^2+bx+c, where a, b, and c are constants and $a \neq 0$; the number a is its **leading coefficient**. The equation $ax^2+bx+c=0$ is a **quadratic equation**. A number that satisfies it is a **root**, or **solution**, of the equation; it is also called a root of the quadratic. In this handout all roots are real numbers.

Proposition 1.1.2 (Zero Product Property)**.** If $pq = 0$, then $p = 0$ or $q = 0$. In particular, if $ax^2 + bx + c = a(x - r)(x - s)$, then the roots of $ax^2 + bx + c = 0$ are r and s, and there are no others.

Proof. If $p \neq 0$, dividing $pq = 0$ by p gives $q = 0$. For the second statement, $a \neq 0$, so $a(x-r)(x-s) = 0$ exactly when $x - r = 0$ or $x - s = 0$. □

Since $(x+m)(x+n) = x^2+(m+n)x+mn$, factoring x^2+bx+c means finding two numbers m and n with $m+n=b$ and $mn=c$. Search the factor pairs of c. If $c>0$, then m and n have the same sign, the sign of b; if $c<0$, they have opposite signs. For example, $3+4=7$ and $3\cdot 4=12$, so $x^2+7x+12=(x+3)(x+4)$, and the roots of $x^2+7x+12=0$ are -3 and -4.

When $a \neq 1$, first take out any common factor of the coefficients. Then find two numbers m and n with $m+n=b$ and $mn=ac$, write bx as $mx+nx$, and factor the four terms in pairs. This method is called **splitting the middle term**. It works because if $ax^2+bx+c=(px+q)(ux+v)$, then the coefficient of x is $pv+qu$, and the two products pv and qu have product $(pu)(qv)=ac$.

An equation that can be written as $Au^2+Bu+C=0$ with $A \neq 0$, where u is an expression in x, is a quadratic equation in u. For example, $x^2=|x|^2$, so an equation $Ax^2+B|x|+C=0$ is a quadratic equation in $u=|x|$. Solve for u, keep only the values that u can take (for example, $u \geq 0$ when u is $|x|$ or a square root), and then solve for x. Squaring both sides of an equation can create values that do not satisfy the original equation, so check every answer in the original equation.

Examples.

Problem 1.1.3. Solve each equation by factoring.

(a) $x^2-5x-14=0$
(b) $6x^2+x-2=0$

Strategy. For (a), find two numbers with product -14 and sum -5. For (b), the leading coefficient is not 1: find two numbers with product $ac = 6\cdot(-2) = -12$ and sum 1, split the middle term, and factor the four terms in pairs.

Solution.

(a) The numbers -7 and 2 have product -14 and sum -5, so

$$x^2-5x-14=(x-7)(x+2).$$

By the Zero Product Property, $x=7$ or $x=-2$.

(b) The numbers 4 and -3 have product -12 and sum 1, so

$$6x^2+x-2=6x^2+4x-3x-2=2x(3x+2)-(3x+2)=(2x-1)(3x+2).$$

Hence $x=\frac{1}{2}$ or $x=-\frac{2}{3}$. Check: $6\cdot\frac{1}{4}+\frac{1}{2}-2=0$ and $6\cdot\frac{4}{9}-\frac{2}{3}-2=0$.

Answer: (a) $x=7$ or $x=-2$; (b) $x=\frac{1}{2}$ or $x=-\frac{2}{3}$. □

Problem 1.1.4 (2011 AMC 10B, Problem #19)**.** What is the product of all the roots of the equation below?

$$\sqrt{5|x|+8} = \sqrt{x^2-16}$$

(A) -64 **(B)** -24 **(C)** -9 **(D)** 24 **(E)** 576

Strategy. Square both sides. Since $x^2 = |x|^2$, the result is a quadratic equation in $|x|$. Keep only the values with $|x| \geq 0$.

Solution. Two square roots are equal exactly when the numbers under them are equal and non-negative. So $5|x| + 8 = x^2 - 16$; both numbers are then at least 8, so both square roots exist. Since $x^2 = |x|^2$, the equation is

$$|x|^2 - 5|x| - 24 = 0, \qquad \text{that is,} \qquad \big(|x| - 8\big)\big(|x| + 3\big) = 0.$$

The factor $|x| + 3$ is positive, so $|x| = 8$, and $x = 8$ or $x = -8$. For both values each side of the original equation equals $\sqrt{48}$. The product of the roots is $8 \cdot (-8) = -64$.

Answer: (A) -64. □

1.2. **Completing the Square.**

Proposition 1.2.1 (Completing the Square)**.** For every number b,

$$x^2 + bx = \left(x + \frac{b}{2}\right)^2 - \frac{b^2}{4}.$$

Proof. Expanding gives $\left(x + \frac{b}{2}\right)^2 = x^2 + bx + \frac{b^2}{4}$. □

For example, $x^2 + 10x = (x+5)^2 - 25$. To solve $x^2 + bx + c = 0$, write it as

$$\left(x + \frac{b}{2}\right)^2 = \frac{b^2}{4} - c,$$

and call the right side d. If $d > 0$, then $x + \frac{b}{2} = \pm\sqrt{d}$; if $d = 0$, then $x = -\frac{b}{2}$; if $d < 0$, there is no real solution, because a square is never negative. To solve $ax^2 + bx + c = 0$, divide by a first.

The square of a real number is never negative, so a sum of squares is 0 only when every square is 0. Completing the square in each unknown can turn an equation in two unknowns into such a sum.

Examples.

Problem 1.2.2. Solve $x^2 - 6x + 2 = 0$ by completing the square.

Strategy. Move the constant term to the right side, then add the square of half the coefficient of x to both sides.

Solution. The equation is $x^2 - 6x = -2$. Half of -6 is -3, and $(-3)^2 = 9$, so adding 9 to both sides gives

$$x^2 - 6x + 9 = 7, \qquad \text{that is,} \qquad (x-3)^2 = 7.$$

Hence $x - 3 = \pm\sqrt{7}$, and $x = 3 \pm \sqrt{7}$.

Answer: $x = 3 + \sqrt{7}$ or $x = 3 - \sqrt{7}$. □

Problem 1.2.3 (2013 AMC 10B, Problem #11)**.** Real numbers x and y satisfy the equation

$$x^2 + y^2 = 10x - 6y - 34.$$

What is $x + y$?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 6 **(E)** 8

Strategy. Gather all the terms on the left, then complete the square in each variable. Squares of real numbers are never negative, so the resulting equation determines both x and y.

Solution. The equation is $x^2 - 10x + y^2 + 6y + 34 = 0$. Since $x^2 - 10x = (x-5)^2 - 25$ and $y^2 + 6y = (y+3)^2 - 9$, it becomes

$$(x-5)^2 + (y+3)^2 - 25 - 9 + 34 = 0, \qquad \text{that is,} \qquad (x-5)^2 + (y+3)^2 = 0.$$

Neither square is negative, so both are 0, which gives $x = 5$ and $y = -3$. Then $x + y = 2$.

Answer: (B) 2. □

1.3. **The Quadratic Formula.**

Theorem 1.3.1 (Quadratic Formula)**.** *Let $a \neq 0$. If $b^2 - 4ac \geq 0$, the solutions of $ax^2 + bx + c = 0$ are*

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

If $b^2 - 4ac < 0$, the equation has no real solutions.

Proof. Divide by a and complete the square:

$$x^2 + \frac{b}{a}x = -\frac{c}{a}, \qquad \text{so} \qquad \left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} = \frac{b^2 - 4ac}{4a^2}.$$

If $b^2 - 4ac \geq 0$, taking square roots gives

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a},$$

and the sign $\pm$ also covers $a < 0$, since $\sqrt{4a^2} = 2|a|$. If $b^2 - 4ac < 0$, the right side is negative and the left side is a square, so there is no solution. □

Before reading off a, b, and c, write the equation in the form $ax^2 + bx + c = 0$, and keep the signs of the coefficients. The formula applies to every quadratic equation; factoring is quicker when the roots are rational, and Section 2.2 says exactly when that happens. For example, for $x^2 - 6x + 2 = 0$ the formula gives

$$x = \frac{6 \pm \sqrt{36 - 8}}{2} = 3 \pm \sqrt{7},$$

the roots found by completing the square in Section 1.2.

Example.

Problem 1.3.2 (2002 AMC 12A, Problem #13)**.** Two different positive numbers a and b each differ from their reciprocals by 1. What is $a + b$?

(A) 1 **(B)** 2 **(C)** $\sqrt{5}$ **(D)** $\sqrt{6}$ **(E)** 3

Strategy. The condition on a number $x > 0$ is that $x - \frac{1}{x} = 1$ or $\frac{1}{x} - x = 1$. Clearing the fraction turns each of these into a quadratic equation; solve both and keep only the positive roots.

Solution. Multiplying by $x > 0$, the two equations become

$$x^2 - x - 1 = 0 \qquad \text{and} \qquad x^2 + x - 1 = 0.$$

By the quadratic formula, their roots are $\frac{1\pm\sqrt{5}}{2}$ and $\frac{-1\pm\sqrt{5}}{2}$, respectively. Since $\sqrt{5} > 1$, each equation has exactly one positive root: $\frac{1+\sqrt{5}}{2}$ and $\frac{-1+\sqrt{5}}{2}$. So exactly two positive numbers differ from their reciprocals by 1, and these are a and b. Their sum is

$$\frac{1+\sqrt{5}}{2} + \frac{-1+\sqrt{5}}{2} = \sqrt{5}.$$

Answer: (C) $\sqrt{5}$. □

2. The Discriminant

The number under the square root in the quadratic formula decides how many real roots a quadratic equation has and, when the coefficients are integers, whether the roots are rational.

2.1. Number of Real Roots.

Definition 2.1.1. The **discriminant** of the quadratic $ax^2 + bx + c$ is $D = b^2 - 4ac$.

Theorem 2.1.2 (Number of Real Roots)**.** *Let $a \neq 0$ and $D = b^2 - 4ac$.*

(a) If $D > 0$, the equation $ax^2 + bx + c = 0$ has two different real roots, $\frac{-b+\sqrt{D}}{2a}$ and $\frac{-b-\sqrt{D}}{2a}$.
(b) If $D = 0$, it has exactly one real root, $x = -\frac{b}{2a}$, and $ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2$.
(c) If $D < 0$, it has no real roots.

Proof. By the proof of the quadratic formula, the equation is equivalent to $\left(x + \frac{b}{2a}\right)^2 = \frac{D}{4a^2}$. A positive number has two square roots, 0 has one, and a negative number has none. For (b), $D = 0$ means $c = \frac{b^2}{4a}$, and $a\left(x + \frac{b}{2a}\right)^2 = ax^2 + bx + \frac{b^2}{4a}$. □

The root in (b) is a **double root**. The wording of a problem translates into a condition on D:

- "only one solution" or "equal roots" means $D = 0$;
- "real roots" means $D \geq 0$, since a double root is a real root;
- "two distinct real roots" means $D > 0$;
- "no real roots" means $D < 0$.

If the coefficient of x^2 contains an unknown, treat separately any value that makes it 0, since the equation is then not quadratic. For example, $x^2 + 2x + 5 = 0$ has $D = 4 - 20 = -16 < 0$, so it has no real roots, as $x^2 + 2x + 5 = (x + 1)^2 + 4 > 0$ also shows.

Examples.

Problem 2.1.3 (2005 AMC 10A, Problem #10)**.** There are two values of a for which the equation $4x^2 + ax + 8x + 9 = 0$ has only one solution for x. What is the sum of those values of a?

(A) -16 **(B)** -8 **(C)** 0 **(D)** 8 **(E)** 20

Strategy. By the theorem on the number of real roots, "only one solution" means $D = 0$, and part (b) then writes the quadratic as $4(x - r)^2$, where r is the solution. Compare coefficients.

Solution. Collecting the terms in x, the equation is $4x^2 + (a + 8)x + 9 = 0$, a quadratic for every a. By the theorem on the number of real roots, it has only one solution r exactly when $D = 0$, and then, by part (b),

$$4x^2 + (a + 8)x + 9 = 4(x - r)^2 = 4x^2 - 8rx + 4r^2.$$

The constant terms give $4r^2 = 9$, so $r = \frac{3}{2}$ or $r = -\frac{3}{2}$. The coefficients of x give $a + 8 = -8r$, which is -12 or 12. So $a = -20$ or $a = 4$, and the sum of the two values is $-20 + 4 = -16$. Check: for both values $(a + 8)^2 = 144 = 4 \cdot 4 \cdot 9$, so $D = 0$.

Answer: (A) -16. □

Problem 2.1.4 (1963 AHSME, Problem #24)**.** Consider equations of the form $x^2 + bx + c = 0$. How many such equations have real roots and have coefficients b and c selected from the set of integers $\{1, 2, 3, 4, 5, 6\}$?

(A) 20 **(B)** 19 **(C)** 18 **(D)** 17 **(E)** 16

Strategy. The equation has real roots exactly when $b^2 - 4c \geq 0$, and a double root counts. For each b, count the values of c with $c \leq \frac{b^2}{4}$.

Solution. For each b, the allowed values of c are the integers from 1 to 6 that are at most $\frac{b^2}{4}$:

b	1	2	3	4	5	6
$\frac{b^2}{4}$	$\frac{1}{4}$	1	$\frac{9}{4}$	4	$\frac{25}{4}$	9
Number of values of c	0	1	2	4	6	6

The total is $0 + 1 + 2 + 4 + 6 + 6 = 19$. The two equations with $D = 0$, namely $x^2 + 2x + 1 = 0$ and $x^2 + 4x + 4 = 0$, are included, because a double root is a real root.

Answer: (B) 19. □

2.2. **Rational Roots.**

Theorem 2.2.1 (Rational Roots)**.** *Let a, b, and c be integers with $a \neq 0$ and $D = b^2 - 4ac \geq 0$. The roots of $ax^2 + bx + c = 0$ are rational exactly when D is a perfect square, that is, the square of an integer.*

Proof. If $D = k^2$ for an integer $k \geq 0$, the roots $\frac{-b \pm k}{2a}$ are rational. Conversely, suppose a root r is rational. By the quadratic formula, $2ar + b = \pm\sqrt{D}$, so $\sqrt{D}$ is rational. The square root of a nonnegative integer is either an integer or irrational, so $\sqrt{D}$ is an integer. □

Corollary 2.2.2. *Let b and c be integers. Then $x^2 + bx + c = (x - r)(x - s)$ for some integers r and s exactly when $b^2 - 4c$ is a perfect square. In that case r and s are the roots of $x^2 + bx + c = 0$.*

Proof. If $x^2 + bx + c = (x - r)(x - s) = x^2 - (r + s)x + rs$, then $b = -(r + s)$ and $c = rs$, so $b^2 - 4c = (r + s)^2 - 4rs = (r - s)^2$. Conversely, let $b^2 - 4c = k^2$ with $k \geq 0$ an integer. Then $b^2 - k^2 = 4c$ is even, so b and k are both even or both odd, and $r = \frac{-b+k}{2}$ and $s = \frac{-b-k}{2}$ are integers. They satisfy $r + s = -b$ and $rs = \frac{b^2-k^2}{4} = c$, so $(x - r)(x - s) = x^2 - (r + s)x + rs = x^2 + bx + c$. □

Example.

Problem 2.2.3 (1989 AHSME, Problem #8)**.** For how many integers n between 1 and 100 does $x^2 + x - n$ factor into the product of two linear factors with integer coefficients?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 9 **(E)** 10

Strategy. By the corollary, $x^2 + x - n$ factors with integer coefficients exactly when its discriminant $1 + 4n$ is a perfect square.

Solution. A factorization $(px + q)(ux + v)$ with integer coefficients has $pu = 1$, so $p = u = 1$ or $p = u = -1$, and in the second case $(-x + q)(-x + v) = (x - q)(x - v)$. So the factorization can be written as $(x - r)(x - s)$ with integers r and s, and by the corollary this happens exactly when $1 + 4n$ is a perfect square, say $1 + 4n = k^2$ with $k \geq 0$ an integer. Then k^2 is odd, so $k = 2j + 1$ for an integer $j \geq 0$, and $1 + 4n = 4j^2 + 4j + 1$ gives

$$n = j(j + 1).$$

For $j = 1, 2, \ldots, 9$ this gives $n = 2, 6, 12, 20, 30, 42, 56, 72, 90$, while $j = 0$ gives $n = 0$ and $j = 10$ gives $n = 110$. So there are 9 such integers; for example, $x^2 + x - 2 = (x + 2)(x - 1)$. Neither 1 nor 100 has the form $j(j + 1)$, so the count is the same whether or not "between" includes them.

Answer: (D) 9. □

3. Vertex Form

Completing the square shows where a quadratic is largest or smallest and what its graph looks like.

3.1. The Vertex and the Axis of Symmetry.

Proposition 3.1.1 (Vertex Form)**.** Let $a \neq 0$ and $f(x) = ax^2 + bx + c$. Then $f(x) = a(x-h)^2 + k$, where

$$h = -\frac{b}{2a} \qquad \text{and} \qquad k = f(h) = c - \frac{b^2}{4a} = -\frac{D}{4a}.$$

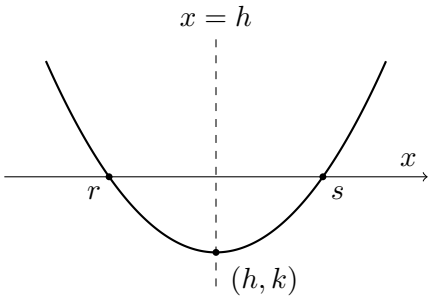
Proof. Completing the square,

$$f(x) = a\left(x^2 + \frac{b}{a}x\right) + c = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c.$$

At $x = h$ the square is 0, so $k = f(h)$, and $c - \frac{b^2}{4a} = -\frac{b^2 - 4ac}{4a}$. □

Definition 3.1.2. The graph of $y = ax^2 + bx + c$ is a **parabola**. It opens upward if $a > 0$ and downward if $a < 0$. Its **vertex** is the point (h, k), and its **axis of symmetry** is the vertical line $x = h$.

The figure shows a parabola that opens upward, with its vertex, its axis of symmetry, and its x-intercepts r and s, the roots of $ax^2 + bx + c = 0$.



Proposition 3.1.3 (Symmetry)**.** Let $f(x) = ax^2 + bx + c = a(x - h)^2 + k$ with $a \neq 0$. For every t, $f(h + t) = f(h - t)$. Consequently:

(a) if $f(p) = f(q)$ and $p \neq q$, then $h = \frac{p+q}{2}$;
(b) if f has two different real roots, they are symmetric about the line $x = h$, so their average is $h = -\frac{b}{2a}$.

Proof. Both $f(h + t)$ and $f(h - t)$ equal $at^2 + k$. For (a), $f(p) = f(q)$ gives $a(p - h)^2 = a(q - h)^2$, so $p - h = \pm(q - h)$. Since $p \neq q$, $p - h = -(q - h)$, that is, $h = \frac{p+q}{2}$. Part (b) is (a) with $f(r) = f(s) = 0$. □

Proposition 3.1.4 (Sign of a Quadratic)**.** Let $a > 0$. If $D > 0$ and the roots of $ax^2 + bx + c = 0$ are $r < s$, then $ax^2 + bx + c < 0$ exactly when $r < x < s$, and $ax^2 + bx + c > 0$ when $x < r$ or $x > s$. If $D < 0$, then $ax^2 + bx + c > 0$ for every x.

Proof. By vertex form, $f(x) = ax^2 + bx + c = a(x-h)^2 - \frac{D}{4a}$. If $D < 0$, then $f(x) \geq -\frac{D}{4a} > 0$. If $D > 0$, then $f(x) < 0$ exactly when $(x-h)^2 < \frac{D}{4a^2}$, that is, when $|x-h| < \frac{\sqrt{D}}{2a}$. By the quadratic formula, the endpoints $h - \frac{\sqrt{D}}{2a}$ and $h + \frac{\sqrt{D}}{2a}$ of this interval are the roots r and s. In the same way, $f(x) > 0$ exactly when $|x-h| > \frac{\sqrt{D}}{2a}$. □

Examples.

Problem 3.1.5 (2006 AMC 10A, Problem #8)**.** A parabola with equation $y = x^2 + bx + c$ passes through the points $(2, 3)$ and $(4, 3)$. What is c?

(A) 2 **(B)** 5 **(C)** 7 **(D)** 10 **(E)** 11

Strategy. With $f(x) = x^2+bx+c$, the two points say that $f(2) = f(4)$, so part (a) of the Symmetry proposition locates the axis of symmetry. The axis determines b, and one point then gives c.

Solution. Let $f(x) = x^2 + bx + c$. Since $f(2) = f(4) = 3$ and $2 \neq 4$, part (a) of the Symmetry proposition gives $h = \frac{2+4}{2} = 3$. By the Vertex Form proposition with $a = 1$, $h = -\frac{b}{2}$, so $b = -2h = -6$. The point $(2, 3)$ then gives $3 = 4 - 12 + c$, so $c = 11$. Check with $(4, 3)$: $16 - 24 + 11 = 3$.

Answer: (E) 11. □

Problem 3.1.6 (1986 AHSME, Problem #13)**.** A parabola $y = ax^2 + bx + c$ has vertex $(4, 2)$. If $(2, 0)$ is on the parabola, then abc equals

(A) -12 **(B)** -6 **(C)** 0 **(D)** 6 **(E)** 12

Strategy. The Vertex Form proposition gives the vertex in terms of a, b, and c. With the vertex $(4, 2)$, this writes b and c in terms of a; then the point $(2, 0)$ gives a.

Solution. By the Vertex Form proposition, the vertex is (h, k) with $h = -\frac{b}{2a}$ and $k = c - \frac{b^2}{4a}$. From $h = 4$, $b = -8a$. From $k = 2$,

$$c = 2 + \frac{b^2}{4a} = 2 + \frac{64a^2}{4a} = 2 + 16a.$$

The point $(2, 0)$ gives $0 = 4a+2b+c = 4a-16a+2+16a = 4a+2$, so $a = -\frac{1}{2}$. Hence $b = -8a = 4$, $c = 2 + 16a = -6$, and $abc = \left(-\frac{1}{2}\right) \cdot 4 \cdot (-6) = 12$.

Answer: (E) 12. □

3.2. **Maximum and Minimum Values.**

Theorem 3.2.1 (Maximum and Minimum)**.** *Let $f(x) = ax^2 + bx + c = a(x-h)^2 + k$. If $a > 0$, the smallest value of f is $k = c - \frac{b^2}{4a}$, taken at $x = h = -\frac{b}{2a}$, and f has no largest value. If $a < 0$, the largest value of f is k, taken at $x = h$, and f has no smallest value.*

Proof. If $a > 0$, then $a(x-h)^2 \geq 0$, with equality only at $x = h$, so $f(x) \geq k = f(h)$. Also $f(x)$ is as large as desired when $|x-h|$ is large. For $a < 0$ the inequalities reverse. □

For example, $2x^2 - 12x + 7 = 2(x^2 - 6x) + 7 = 2(x-3)^2 - 11$, so its smallest value is -11, at $x = 3$.

To find the largest or smallest value of a quantity, write it as a quadratic in one variable, using the conditions of the problem to remove the other variables. Check that the best value of the variable is allowed (for example, a length between 0 and 1), and then read off k. When the quantity is a sum of quadratics in variables that can change independently, complete the square in each of them.

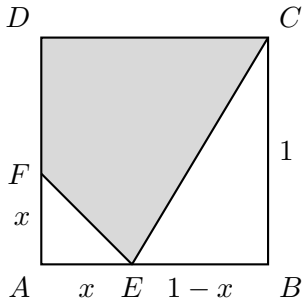
Example.

Problem 3.2.2 (1962 AHSME, Problem #37)**.** $ABCD$ is a square with side of unit length. Points E and F are taken respectively on sides AB and AD so that $AE = AF$ and the quadrilateral $CDFE$ has maximum area. In square units this maximum area is:

(A) $\frac{1}{2}$ **(B)** $\frac{9}{16}$ **(C)** $\frac{19}{32}$ **(D)** $\frac{5}{8}$ **(E)** $\frac{2}{3}$

Strategy. Let $AE = AF = x$. The quadrilateral $CDFE$ is the square with triangles AEF and EBC removed, so its area is a quadratic in x. Find its largest value for $0 \le x \le 1$.

Solution. In the figure, $AE = AF = x$, and the shaded region is $CDFE$.



Triangle AEF has a right angle at A and legs $AE = AF = x$, so its area is $\frac{x^2}{2}$. Triangle EBC has a right angle at B and legs $EB = 1 - x$ and $BC = 1$, so its area is $\frac{1-x}{2}$. Hence the area of $CDFE$ is

$$1 - \frac{x^2}{2} - \frac{1-x}{2} = -\frac{1}{2}x^2 + \frac{1}{2}x + \frac{1}{2} = -\frac{1}{2}\left(x - \frac{1}{2}\right)^2 + \frac{5}{8}.$$

The leading coefficient is negative, so the largest value is $\frac{5}{8}$, taken at $x = \frac{1}{2}$, which is allowed since $0 \le \frac{1}{2} \le 1$.

Answer: (D) $\frac{5}{8}$. □

4. Vieta's Formulas

The sum and the product of the roots of a quadratic can be read from its coefficients, without solving the equation.

4.1. Sum and Product of the Roots.

Theorem 4.1.1 (Vieta's Formulas)**.** *Let $a \neq 0$ and $D \ge 0$, and let r and s be the roots of $ax^2 + bx + c = 0$, with $r = s = -\frac{b}{2a}$ when $D = 0$. Then*

$$r + s = -\frac{b}{a}, \qquad rs = \frac{c}{a}, \qquad \text{and} \qquad ax^2 + bx + c = a(x - r)(x - s).$$

Proof. By the quadratic formula, r and s are $\frac{-b+\sqrt{D}}{2a}$ and $\frac{-b-\sqrt{D}}{2a}$. Adding them, the square roots cancel, so $r + s = \frac{-2b}{2a} = -\frac{b}{a}$. Multiplying them,

$$rs = \frac{(-b)^2 - D}{4a^2} = \frac{b^2 - (b^2 - 4ac)}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}.$$

Finally, $a(x - r)(x - s) = a\big(x^2 - (r + s)x + rs\big) = ax^2 + bx + c$. □

For $x^2 + bx + c$, the sum of the roots is $-b$ and the product is c. In this handout the formulas are used only for real roots, so check that $D \ge 0$ when it is not given.

Corollary 4.1.2. *If two numbers have sum S and product P, they are the roots of $x^2-Sx+P=0$. Real numbers with sum S and product P exist exactly when $S^2-4P\geq 0$.*

Proof. If $r+s=S$ and $rs=P$, then $(x-r)(x-s)=x^2-(r+s)x+rs=x^2-Sx+P$, so r and s are the roots of $x^2-Sx+P=0$ by the Zero Product Property. By the theorem on the number of real roots, this equation has real roots exactly when its discriminant S^2-4P is at least 0, and then its two roots (equal when $S^2-4P=0$) have sum S and product P by Vieta's formulas. □

Example.

Problem 4.1.3 (1953 AHSME, Problem #44)**.** In solving a problem that reduces to a quadratic equation one student makes a mistake only in the constant term of the equation and obtains 8 and 2 for the roots. Another student makes a mistake only in the coefficient of the first degree term and finds -9 and -1 for the roots. The correct equation was:
(A) $x^2-10x+9=0$ **(B)** $x^2+10x+9=0$ **(C)** $x^2-10x+16=0$
(D) $x^2-8x-9=0$ **(E)** none of these

Strategy. A mistake only in the constant term does not change the sum of the roots, $-\frac{b}{a}$. A mistake only in the coefficient of x does not change the product of the roots, $\frac{c}{a}$.

Solution. Divide the correct equation by its leading coefficient to write it as $x^2+bx+c=0$; every choice already has this form. The first student's equation is $x^2+bx+c'=0$ with the correct b, so by Vieta's formulas $-b=8+2=10$. The second student's equation is $x^2+b'x+c=0$ with the correct c, so $c=(-9)\cdot(-1)=9$. The correct equation is $x^2-10x+9=0$. Check: with the constant term changed to 16, it becomes $x^2-10x+16=(x-2)(x-8)=0$, with roots 8 and 2; with the coefficient of x changed to 10, it becomes $x^2+10x+9=(x+1)(x+9)=0$, with roots -9 and -1.

Answer: (A) $x^2-10x+9=0$. □

4.2. **Expressions in the Roots.**

Proposition 4.2.1 (Expressions in the Roots)**.** If r and s are the roots of $ax^2+bx+c=0$, where $D\geq 0$, then

$$r^2+s^2=(r+s)^2-2rs \qquad \text{and} \qquad \frac{1}{r}+\frac{1}{s}=\frac{r+s}{rs} \quad (rs\neq 0),$$
$$(r-s)^2=(r+s)^2-4rs=\frac{D}{a^2}, \qquad \text{so} \qquad |r-s|=\frac{\sqrt{D}}{|a|}.$$

Proof. The first identity follows from $(r+s)^2=r^2+2rs+s^2$, and the second from combining the fractions. For the third, $(r+s)^2-4rs=r^2-2rs+s^2=(r-s)^2$, and by Vieta's formulas

$$(r+s)^2-4rs=\frac{b^2}{a^2}-\frac{4c}{a}=\frac{b^2-4ac}{a^2}.$$ □

To find a quadratic whose roots are two expressions u and v in r and s, such as $r+\frac{1}{s}$ and $s+\frac{1}{r}$, compute $u+v$ and uv from Vieta's formulas. By the corollary in Section 4.1, the quadratic $x^2-(u+v)x+uv$ has roots u and v.

The identities hold for any two numbers whose sum S and product P are known, even when no quadratic is mentioned. By the corollary in Section 4.1, such numbers are the roots of $t^2-St+P=0$, and they are real exactly when $S^2-4P\geq 0$.

Examples.

Problem 4.2.2. The roots of $x^2 - 5x + 3 = 0$ are r and s. Without solving the equation, find $r^2 + s^2$, $\frac{1}{r} + \frac{1}{s}$, and $|r - s|$.

Strategy. Find $r + s$ and rs from Vieta's formulas and use the three identities.

Solution. The discriminant is $25 - 12 = 13 > 0$, so the roots are real and different. By Vieta's formulas, $r + s = 5$ and $rs = 3$. Then

$$r^2 + s^2 = (r + s)^2 - 2rs = 25 - 6 = 19 \qquad \text{and} \qquad \frac{1}{r} + \frac{1}{s} = \frac{r + s}{rs} = \frac{5}{3}.$$

Also $(r - s)^2 = (r + s)^2 - 4rs = 25 - 12 = 13$, so $|r - s| = \sqrt{13}$, which is $\frac{\sqrt{D}}{|a|}$ with $D = 13$ and $a = 1$. As a check, the quadratic formula gives the roots $\frac{5 \pm \sqrt{13}}{2}$, whose difference is $\sqrt{13}$.

Answer: $r^2 + s^2 = 19$, $\frac{1}{r} + \frac{1}{s} = \frac{5}{3}$, and $|r - s| = \sqrt{13}$. □

Problem 4.2.3 (2006 AMC 10B, Problem #14)**.** Let a and b be the roots of the equation $x^2 - mx + 2 = 0$. Suppose that $a + \frac{1}{b}$ and $b + \frac{1}{a}$ are the roots of the equation $x^2 - px + q = 0$. What is q?

(A) $\frac{5}{2}$ **(B)** $\frac{7}{2}$ **(C)** 4 **(D)** $\frac{9}{2}$ **(E)** 8

Strategy. Write $a + \frac{1}{b}$ and $b + \frac{1}{a}$ as single fractions. Their product is q by Vieta's formulas, and it depends only on ab, which the first equation gives.

Solution. By Vieta's formulas for the first equation, $ab = 2$ (and $a + b = m$; the roots are real when $m^2 \geq 8$). So

$$a + \frac{1}{b} = \frac{ab + 1}{b} = \frac{3}{b} \qquad \text{and} \qquad b + \frac{1}{a} = \frac{ab + 1}{a} = \frac{3}{a}.$$

These are the roots of the second equation, so by Vieta's formulas

$$q = \frac{3}{b} \cdot \frac{3}{a} = \frac{9}{ab} = \frac{9}{2}.$$

The value of q does not depend on m.

Answer: (D) $\frac{9}{2}$. □

Problem 4.2.4 (2013 AMC 12B, Problem #17)**.** Let a, b, and c be real numbers such that

$$a + b + c = 2 \qquad \text{and} \qquad a^2 + b^2 + c^2 = 12.$$

What is the difference between the maximum and minimum possible values of c?

(A) 2 **(B)** $\frac{10}{3}$ **(C)** 4 **(D)** $\frac{16}{3}$ **(E)** $\frac{20}{3}$

Strategy. Fix c. The conditions give $a + b$ and $a^2 + b^2$, hence ab. Then a and b are the roots of a quadratic, and real a and b exist exactly when its discriminant is at least 0. This is a quadratic inequality in c.

Solution. Moving c and c^2 to the right sides gives $a + b = 2 - c$ and $a^2 + b^2 = 12 - c^2$. By the first identity of the proposition,

$$2ab = (a + b)^2 - (a^2 + b^2) = (2 - c)^2 - (12 - c^2) = 2c^2 - 4c - 8,$$

so $ab = c^2 - 2c - 4$. By the corollary in Section 4.1, real numbers a and b with this sum and product exist exactly when

$$(2-c)^2 - 4(c^2 - 2c - 4) = -3c^2 + 4c + 20 \geq 0, \qquad \text{that is,} \qquad 3c^2 - 4c - 20 \leq 0.$$

The discriminant of $3c^2 - 4c - 20$ is $16 + 240 = 256 = 16^2$, a perfect square, so by the Rational Roots theorem its roots are rational. Indeed, splitting the middle term with $-10 + 6 = -4$ and $(-10) \cdot 6 = 3 \cdot (-20)$ gives

$$3c^2 - 4c - 20 = 3c^2 - 10c + 6c - 20 = c(3c - 10) + 2(3c - 10) = (3c - 10)(c + 2),$$

so the roots are -2 and $\frac{10}{3}$. The leading coefficient is positive, so by the Sign of a Quadratic, $3c^2 - 4c - 20 \leq 0$ exactly when $-2 \leq c \leq \frac{10}{3}$. Both ends occur: $c = \frac{10}{3}$ with $a = b = -\frac{2}{3}$, and $c = -2$ with $a = b = 2$. The difference between the largest and smallest values of c is

$$\frac{10}{3} - (-2) = \frac{16}{3}.$$

This is the difference of the roots of $3c^2 - 4c - 20$, which by the proposition is $\frac{\sqrt{16+240}}{3} = \frac{16}{3}$.

Answer: (D) $\dfrac{16}{3}$. □

4.3. **Integer Roots.** If $x^2 + bx + c$ has integer roots r and s, then $rs = c$ and $r + s = -b$. So r and s form a factor pair of c, where negative factors are allowed, and each such pair gives one possible value $b = -(r + s)$. Two more facts help.

- The roots of a quadratic with integer coefficients can be integers only if D is a perfect square (Section 2.2).
- If a prime p divides a product rs of integers, then p divides r or s.

Examples.

Problem 4.3.1 (2022 AMC 10B, Problem #7)**.** For how many values of the constant k will the polynomial $x^2 + kx + 36$ have two distinct integer roots?

(A) 6 **(B)** 8 **(C)** 9 **(D)** 14 **(E)** 16

Strategy. By Vieta's formulas, the two roots have product 36 and sum $-k$. So list the pairs of different integers with product 36, and check that no two pairs have the same sum.

Solution. Call the roots r and s, with $r \neq s$. By Vieta's formulas, $r + s = -k$ and $rs = 36$. The pairs of different integers with product 36 are $\{1, 36\}$, $\{2, 18\}$, $\{3, 12\}$, $\{4, 9\}$, and the four pairs of their negatives; $6 \cdot 6$ and $(-6) \cdot (-6)$ do not count, since they would make the two roots equal. The sums are ± 37, ± 20, ± 15, and ± 13, eight different values, and for each of them $x^2 + kx + 36 = (x - r)(x - s)$ has the two distinct integer roots r and s. So there are 8 values of k.

Answer: (B) 8. □

Problem 4.3.2 (1987 AHSME, Problem #23)**.** If p is a prime and both roots of $x^2 + px - 444p = 0$ are integers, then

(A) $1 < p \leq 11$ **(B)** $11 < p \leq 21$ **(C)** $21 < p \leq 31$ **(D)** $31 < p \leq 41$
(E) $41 < p \leq 51$

Strategy. Let the integer roots be r and s, so $r + s = -p$ and $rs = -444p$. The prime p divides rs, so it divides r or s; the sum then shows that it divides both.

Solution. By Vieta's formulas, $r + s = -p$ and $rs = -444p$. Since p is a prime that divides rs, it divides r or s, and since $r + s = -p$, it then divides both. Write $r = pu$ and $s = pv$ with integers u and v. Then $u + v = -1$ and $p^2uv = -444p$, so $puv = -444$. Hence p divides $444 = 2^2 \cdot 3 \cdot 37$, and p is 2, 3, or 37. Also $v = -1 - u$, so $uv = -u(u + 1)$ and

$$u(u + 1) = \frac{444}{p},$$

which is 222, 148, or 12. Replacing u by $-1 - u$ does not change $u(u + 1)$, so it is enough to try $u \geq 0$, where $u(u + 1)$ increases with u. Since $11 \cdot 12 = 132$, $12 \cdot 13 = 156$, $14 \cdot 15 = 210$, and $15 \cdot 16 = 240$, neither 148 nor 222 is of this form, while $12 = 3 \cdot 4$. So $p = 37$, and $u = 3$ gives the roots 111 and -148. Indeed $111 + (-148) = -37$ and $111 \cdot (-148) = -444 \cdot 37$. Since $31 < 37 \leq 41$, the answer is (D).

The discriminant gives a second solution. By the corollary in Section 2.2, both roots are integers exactly when $D = p^2 + 4 \cdot 444p = p(p + 1776)$ is a perfect square k^2. Then p divides k^2, so p divides k, and p^2 divides $k^2 = p(p+1776)$. Hence p divides $p+1776$, so p divides $1776 = 2^4 \cdot 3 \cdot 37$. For $p = 2$ and $p = 3$ the discriminant is 3556 and 5337, which are not perfect squares ($59^2 < 3556 < 60^2$ and $73^2 < 5337 < 74^2$), while $p = 37$ gives $37^2 \cdot 49 = 259^2$.

Answer: (D) $31 < p \leq 41$. □

5. Problems

Beginner.

1. (2008 AMC 10B, Problem #9) A quadratic equation $ax^2 - 2ax + b = 0$ has two real solutions. What is the average of the solutions?

(A) 1 **(B)** 2 **(C)** $\frac{b}{a}$ **(D)** $\frac{2b}{a}$ **(E)** $\sqrt{2a-b}$

2. (2003 AMC 10A, Problem #5) Let d and e denote the solutions of $2x^2+3x-5=0$. What is the value of $(d-1)(e-1)$?

(A) $-\frac{5}{2}$ **(B)** 0 **(C)** 3 **(D)** 5 **(E)** 6

3. (1963 AHSME, Problem #29) A particle projected vertically upward reaches, at the end of t seconds, an elevation of s feet where $s = 160t - 16t^2$. The highest elevation is:

(A) 800 **(B)** 640 **(C)** 400 **(D)** 320 **(E)** 160

4. (2020 AMC 10A, Problem #5) What is the sum of all real numbers x for which

$$|x^2 - 12x + 34| = 2?$$

(A) 12 **(B)** 15 **(C)** 18 **(D)** 21 **(E)** 25

5. (2015 AMC 10A, Problem #12) Points $(\sqrt{\pi}, a)$ and $(\sqrt{\pi}, b)$ are distinct points on the graph of

$$y^2 + x^4 = 2x^2y + 1.$$

What is $|a-b|$?

(A) 1 **(B)** $\frac{\pi}{2}$ **(C)** 2 **(D)** $\sqrt{1+\pi}$ **(E)** $1+\sqrt{\pi}$

Intermediate.

6. (1987 AHSME, Problem #15) If (x, y) is a solution to the system

$$xy = 6,$$
$$x^2y + xy^2 + x + y = 63,$$

find $x^2 + y^2$.

(A) 13 **(B)** $\frac{1173}{32}$ **(C)** 55 **(D)** 69 **(E)** 81

7. (2006 AMC 12B, Problem #12) The parabola $y = ax^2 + bx + c$ has vertex (p, p) and y-intercept $(0, -p)$, where $p \neq 0$. What is b?

(A) $-p$ **(B)** 0 **(C)** 2 **(D)** 4 **(E)** p

8. (1974 AHSME, Problem #10) What is the smallest integral value of k such that

$$2x(kx-4) - x^2 + 6 = 0$$

has no real roots?

(A) -1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

9. (2024 AMC 12B, Problem #13) There are real numbers x, y, h, and k that satisfy the system of equations

$$x^2 + y^2 - 6x - 8y = h$$
$$x^2 + y^2 - 10x + 4y = k.$$

What is the minimum possible value of $h + k$?

(A) -54 **(B)** -46 **(C)** -34 **(D)** -16 **(E)** 16

10. (2000 AMC 10, Problem #24) Let f be a function for which $f\left(\frac{x}{3}\right) = x^2 + x + 1$. Find the sum of all values of z for which $f(3z) = 7$.

(A) $-\frac{1}{3}$ **(B)** $-\frac{1}{9}$ **(C)** 0 **(D)** $\frac{5}{9}$ **(E)** $\frac{5}{3}$

Advanced.

11. (1975 AHSME, Problem #22) If p and q are primes and $x^2 - px + q = 0$ has distinct positive integral roots, then which of the following statements are true?
I. The difference of the roots is odd.
II. At least one root is prime.
III. $p^2 - q$ is prime.
IV. $p + q$ is prime.

(A) I only **(B)** II only **(C)** II and III only **(D)** I, II and IV only **(E)** All are true.

12. (2021 Fall AMC 10A, Problem #20) For how many ordered pairs (b, c) of positive integers does neither $x^2 + bx + c = 0$ nor $x^2 + cx + b = 0$ have two distinct real solutions?

(A) 4 **(B)** 6 **(C)** 8 **(D)** 12 **(E)** 16

13. (2014 AMC 12B, Problem #17) Let P be the parabola with equation $y = x^2$ and let $Q = (20, 14)$. There are real numbers r and s such that the line through Q with slope m does not intersect P if and only if $r < m < s$. What is $r + s$?

(A) 1 **(B)** 26 **(C)** 40 **(D)** 52 **(E)** 80

14. (1983 AIME, Problem #3) What is the product of the real roots of the equation

$$x^2 + 18x + 30 = 2\sqrt{x^2 + 18x + 45}?$$

15. (2014 AIME I, Problem #6) The graphs $y = 3(x - h)^2 + j$ and $y = 2(x - h)^2 + k$ have y-intercepts of 2013 and 2014, respectively, and each graph has two positive integer x-intercepts. Find h.