QUADRATICS, THE DISCRIMINANT, AND VIETA'S FORMULAS FOR AMC 10: SOLUTIONS

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1. Solutions

Beginner.

1. (2008 AMC 10B, Problem #9) A quadratic equation $ax^2 - 2ax + b = 0$ has two real solutions. What is the average of the solutions?

(A) 1 **(B)** 2 **(C)** $\frac{b}{a}$ **(D)** $\frac{2b}{a}$ **(E)** $\sqrt{2a-b}$

Strategy. The average of the solutions is half their sum, and Vieta's formulas give the sum.

Solution. The solutions are real, so Vieta's formulas apply. Their sum is

$$-\frac{-2a}{a} = 2,$$

so their average is $\frac{2}{2} = 1$. Equivalently, the two solutions are symmetric about the axis of symmetry of the parabola $y = ax^2 - 2ax + b$, which is the line $x = -\frac{-2a}{2a} = 1$, so their average is 1.

Answer: (A) 1. □

2. (2003 AMC 10A, Problem #5) Let d and e denote the solutions of $2x^2 + 3x - 5 = 0$. What is the value of $(d-1)(e-1)$?

(A) $-\frac{5}{2}$ **(B)** 0 **(C)** 3 **(D)** 5 **(E)** 6

Strategy. Factor by splitting the middle term, or expand $(d-1)(e-1)$ and use Vieta's formulas.

Solution. Here $ac = 2 \cdot (-5) = -10$, and 5 and -2 have product -10 and sum 3, so

$$2x^2 + 3x - 5 = 2x^2 + 5x - 2x - 5 = x(2x+5) - (2x+5) = (x-1)(2x+5).$$

So one of the solutions is 1, and $(d-1)(e-1) = 0$.

Vieta's formulas give the same result. The discriminant $9 + 40 = 49$ is positive, $d + e = -\frac{3}{2}$, and $de = -\frac{5}{2}$, so

$$(d-1)(e-1) = de - (d+e) + 1 = -\frac{5}{2} + \frac{3}{2} + 1 = 0.$$

Answer: (B) 0. □

3. (1963 AHSME, Problem #29) A particle projected vertically upward reaches, at the end of t seconds, an elevation of s feet where $s = 160t - 16t^2$. The highest elevation is:

(A) 800 **(B)** 640 **(C)** 400 **(D)** 320 **(E)** 160

Strategy. The elevation s is a quadratic in t with a negative leading coefficient, so its largest value is at the vertex.

Solution. Completing the square,

$$s = -16(t^2 - 10t) = -16\big((t-5)^2 - 25\big) = -16(t-5)^2 + 400.$$

Since $-16(t-5)^2 \le 0$, with equality at $t = 5$, the highest elevation is 400 feet, reached after 5 seconds.

Answer: (C) 400. □

4. (2020 AMC 10A, Problem #5) What is the sum of all real numbers x for which

$$|x^2 - 12x + 34| = 2?$$

(A) 12 **(B)** 15 **(C)** 18 **(D)** 21 **(E)** 25

Strategy. The equation $|u| = 2$ means $u = 2$ or $u = -2$. Solve both quadratic equations and add the different solutions.

Solution. The equation $x^2-12x+34 = 2$ is $x^2-12x+32 = 0$, that is, $(x-4)(x-8) = 0$, so $x = 4$ or $x = 8$. The equation $x^2-12x+34 = -2$ is $x^2-12x+36 = 0$, that is, $(x-6)^2 = 0$, so $x = 6$, a double root. The real numbers x that satisfy the original equation are 4, 6, and 8, and their sum is 18. The double root 6 is one number, so it is added once.

Answer: (C) 18. □

5. (2015 AMC 10A, Problem #12) Points $(\sqrt{\pi}, a)$ and $(\sqrt{\pi}, b)$ are distinct points on the graph of

$$y^2 + x^4 = 2x^2y + 1.$$

What is $|a - b|$?

(A) 1 **(B)** $\frac{\pi}{2}$ **(C)** 2 **(D)** $\sqrt{1+\pi}$ **(E)** $1+\sqrt{\pi}$

Strategy. Move $2x^2y$ to the left side and complete the square in y.

Solution. The equation is $y^2 - 2x^2y + x^4 = 1$. Completing the square in y, the left side is $(y - x^2)^2$, so

$$(y - x^2)^2 = 1, \qquad \text{that is,} \qquad y = x^2 + 1 \quad \text{or} \quad y = x^2 - 1.$$

At $x = \sqrt{\pi}$, $x^2 = \pi$, so the two points have y-coordinates $\pi + 1$ and $\pi - 1$. These are a and b in some order, and $|a - b| = 2$.

Answer: (C) 2. □

Intermediate.

6. (1987 AHSME, Problem #15) If (x, y) is a solution to the system

$$xy = 6,$$
$$x^2y + xy^2 + x + y = 63,$$

find $x^2 + y^2$.

(A) 13 **(B)** $\frac{1173}{32}$ **(C)** 55 **(D)** 69 **(E)** 81

Strategy. Write the second equation in terms of the sum $S = x+y$ and the product $xy = 6$; this gives S. Then $x^2 + y^2$ follows from S and xy, as in Section 4.2 of the handout.

Solution. Let $S = x + y$. Since $x^2y + xy^2 = xy \cdot S = 6S$, the second equation reads $6S + S = 63$, so $S = 9$. As in Section 4.2 of the handout,

$$x^2 + y^2 = S^2 - 2xy = 81 - 12 = 69.$$

Real solutions exist: by the corollary in Section 4.1 of the handout, x and y are the roots of $t^2 - 9t + 6 = 0$, whose discriminant $81 - 24 = 57$ is positive.

Answer: (D) 69. □

7. (2006 AMC 12B, Problem #12) The parabola $y = ax^2 + bx + c$ has vertex (p, p) and y-intercept $(0, -p)$, where $p \neq 0$. What is b?

(A) $-p$ **(B)** 0 **(C)** 2 **(D)** 4 **(E)** p

Strategy. By the Vertex Form proposition, the vertex gives $-\frac{b}{2a} = p$ and $c - \frac{b^2}{4a} = p$, and the y-intercept gives $c = -p$. Solve these equations for b.

Solution. The y-intercept is the point $(0, c)$, so $c = -p$. By the Vertex Form proposition, the vertex is (h, k) with $h = -\frac{b}{2a}$ and $k = c - \frac{b^2}{4a}$. From $h = p$, $b = -2ap$. Then

$$k = c - \frac{b^2}{4a} = -p - \frac{4a^2p^2}{4a} = -p - ap^2,$$

and $k = p$ gives $ap^2 = -2p$. Since $p \neq 0$, $ap = -2$, so $b = -2ap = 4$.

Answer: (D) 4. □

8. (1974 AHSME, Problem #10) What is the smallest integral value of k such that

$$2x(kx - 4) - x^2 + 6 = 0$$

has no real roots?

(A) -1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

Strategy. Expand the left side into the form $Ax^2 + Bx + C$, check that the equation is quadratic, and require a negative discriminant.

Solution. Expanding, the equation is $2kx^2 - 8x - x^2 + 6 = 0$, that is, $(2k-1)x^2 - 8x + 6 = 0$. For an integer k, $2k - 1 \neq 0$, so the equation is quadratic. It has no real roots exactly when

$$D = (-8)^2 - 4 \cdot (2k - 1) \cdot 6 = 64 - 24(2k - 1) < 0,$$

that is, $2k - 1 > \frac{64}{24} = \frac{8}{3}$, or $k > \frac{11}{6}$. The smallest such integer is $k = 2$. Check: $k = 2$ gives $3x^2 - 8x + 6 = 0$, with $D = 64 - 72 = -8 < 0$, while $k = 1$ gives $x^2 - 8x + 6 = 0$, with $D = 64 - 24 = 40 > 0$.

Answer: (B) 2. □

9. (2024 AMC 12B, Problem #13) There are real numbers x, y, h, and k that satisfy the system of equations

$$x^2 + y^2 - 6x - 8y = h$$
$$x^2 + y^2 - 10x + 4y = k.$$

What is the minimum possible value of $h + k$?

(A) -54 **(B)** -46 **(C)** -34 **(D)** -16 **(E)** 16

Strategy. For any real x and y, the equations define h and k. Add the two equations and complete the square in x and in y.

Solution. Every choice of real numbers x and y gives values of h and k that satisfy the system. Adding the two equations gives $h + k = 2x^2 + 2y^2 - 16x - 4y$. Completing the square in x and in y,

$$h + k = 2(x^2 - 8x) + 2(y^2 - 2y) = 2(x - 4)^2 - 32 + 2(y - 1)^2 - 2,$$

so $h + k = 2(x - 4)^2 + 2(y - 1)^2 - 34 \geq -34$, with equality at $x = 4$ and $y = 1$; then $h = -15$ and $k = -19$. The minimum possible value of $h + k$ is -34.

Answer: (C) -34. □

10. (2000 AMC 10, Problem #24) Let f be a function for which $f\left(\frac{x}{3}\right) = x^2 + x + 1$. Find the sum of all values of z for which $f(3z) = 7$.

(A) $-\frac{1}{3}$ **(B)** $-\frac{1}{9}$ **(C)** 0 **(D)** $\frac{5}{9}$ **(E)** $\frac{5}{3}$

Strategy. First find $f(t)$ for every t by putting $x = 3t$ in the definition. Then $f(3z) = 7$ is a quadratic equation in z; check that its roots are real and add them with Vieta's formulas.

Solution. Putting $x = 3t$ in the definition gives $f(t) = (3t)^2 + 3t + 1 = 9t^2 + 3t + 1$ for every t. So

$$f(3z) = 9(3z)^2 + 3(3z) + 1 = 81z^2 + 9z + 1,$$

and $f(3z) = 7$ becomes $27z^2 + 3z - 2 = 0$ after subtracting 7 and dividing by 3. Its discriminant $3^2 + 4 \cdot 27 \cdot 2 = 225$ is positive, so it has two real roots, and by Vieta's formulas their sum is $-\frac{3}{27} = -\frac{1}{9}$. (The roots are $\frac{2}{9}$ and $-\frac{1}{3}$.)

Answer: (B) $-\dfrac{1}{9}$. □

Advanced.

11. (1975 AHSME, Problem #22) If p and q are primes and $x^2 - px + q = 0$ has distinct positive integral roots, then which of the following statements are true?
I. The difference of the roots is odd.
II. At least one root is prime.
III. $p^2 - q$ is prime.
IV. $p + q$ is prime.
(A) I only **(B)** II only **(C)** II and III only **(D)** I, II and IV only
(E) All are true.

Strategy. By Vieta's formulas the roots have sum p and product q. A prime is a product of two different positive integers in only one way, which gives the roots in terms of q; then the condition that p is prime determines q.

Solution. Let the roots be the positive integers $r < s$. By Vieta's formulas, $r + s = p$ and $rs = q$. Since q is prime, its only factorization into two different positive integers is $1 \cdot q$, so $r = 1$ and $s = q$, and $p = q + 1$. The primes q and $q + 1$ are consecutive integers, so one of them is even, and therefore equal to 2, the only even prime. It is not $q + 1$, since $q \geq 2$ gives $q + 1 \geq 3$. Hence $q = 2$ and $p = 3$, and the equation is $x^2 - 3x + 2 = 0$, with roots 1 and 2. Now check the statements.

- I: the difference of the roots is $2 - 1 = 1$, which is odd.
- II: the root 2 is prime.
- III: $p^2 - q = 9 - 2 = 7$ is prime.
- IV: $p + q = 3 + 2 = 5$ is prime.

All four statements are true.

Answer: (E) All are true. □

12. (2021 Fall AMC 10A, Problem #20) For how many ordered pairs (b, c) of positive integers does neither $x^2 + bx + c = 0$ nor $x^2 + cx + b = 0$ have two distinct real solutions?
(A) 4 **(B)** 6 **(C)** 8 **(D)** 12 **(E)** 16

Strategy. By the theorem on the number of real roots, the condition is $b^2 \leq 4c$ and $c^2 \leq 4b$. If m is the larger of b and c, one of these inequalities gives $m^2 \leq 4m$, so b and c are at most 4. Then check the remaining pairs.

Solution. By the theorem on the number of real roots, each equation must have $D \leq 0$. So the pairs to count are those with

$$b^2 \leq 4c \qquad \text{and} \qquad c^2 \leq 4b.$$

Let m be the larger of b and c (either one if they are equal). If $m = c$, then $c^2 \le 4b \le 4c$; if $m = b$, then $b^2 \le 4c \le 4b$. Either way $m^2 \le 4m$, so $m \le 4$, and both b and c are at most 4. For each b from 1 to 4, the conditions $c \ge \frac{b^2}{4}$ and $c^2 \le 4b$ give:

- $b = 1$: $c \ge \frac{1}{4}$ and $c^2 \le 4$, so $c = 1$ or $c = 2$;
- $b = 2$: $c \ge 1$ and $c^2 \le 8$, so $c = 1$ or $c = 2$;
- $b = 3$: $c \ge \frac{9}{4}$ and $c^2 \le 12$, so $c = 3$;
- $b = 4$: $c \ge 4$ and $c^2 \le 16$, so $c = 4$.

The six pairs are $(1, 1)$, $(1, 2)$, $(2, 1)$, $(2, 2)$, $(3, 3)$, and $(4, 4)$.

Answer: (B) 6. □

13. (2014 AMC 12B, Problem #17) Let P be the parabola with equation $y = x^2$ and let $Q = (20, 14)$. There are real numbers r and s such that the line through Q with slope m does not intersect P if and only if $r < m < s$. What is $r + s$?

(A) 1 **(B)** 26 **(C)** 40 **(D)** 52 **(E)** 80

Strategy. The line misses the parabola exactly when a certain quadratic equation in x has no real solution. Its discriminant is a quadratic in m, and r and s are the roots of that quadratic.

Solution. A point of P has the form (x, x^2), and it lies on the line through Q with slope m exactly when $x^2 - 14 = m(x - 20)$, that is, when

$$x^2 - mx + 20m - 14 = 0.$$

The line misses P when no real x satisfies this equation, which by the theorem on the number of real roots happens exactly when its discriminant is negative:

$$m^2 - 4(20m - 14) = m^2 - 80m + 56 < 0.$$

As a quadratic in m, $m^2 - 80m + 56$ has a positive leading coefficient and discriminant $80^2 - 4 \cdot 56 = 6176 > 0$, so by the Sign of a Quadratic it is negative exactly when m is strictly between its two roots. These roots are therefore r and s, and by part (b) of the Symmetry proposition their average is $-\frac{-80}{2} = 40$. Hence $r + s = 80$. (Explicitly, $r = 40 - \sqrt{1544}$ and $s = 40 + \sqrt{1544}$.)

Answer: (E) 80. □

14. (1983 AIME, Problem #3) What is the product of the real roots of the equation

$$x^2 + 18x + 30 = 2\sqrt{x^2 + 18x + 45}?$$

Strategy. Both sides depend on x only through $u = x^2 + 18x$. Squaring gives a quadratic equation in u; keep only the values of u that satisfy the equation before squaring, and then apply Vieta's formulas to the equation for x.

Solution. Let $u = x^2 + 18x$, so the equation is $u + 30 = 2\sqrt{u + 45}$. Squaring both sides gives

$$(u + 30)^2 = 4(u + 45), \qquad \text{that is,} \qquad u^2 + 56u + 720 = (u + 20)(u + 36) = 0.$$

Squaring can create values that do not satisfy the original equation, so check both. For $u = -36$, the left side $u + 30$ is -6 and the right side is $2\sqrt{9} = 6$, so $u = -36$ fails. For $u = -20$, both sides equal 10. So the real roots of the original equation are the real solutions of $x^2 + 18x = -20$, that is, of

$$x^2 + 18x + 20 = 0.$$

Its discriminant $18^2 - 4 \cdot 20 = 244$ is positive, so it has two real roots, and Vieta's formulas give their product as 20.

Answer: 20. □

15. (2014 AIME I, Problem #6) The graphs $y = 3(x-h)^2 + j$ and $y = 2(x-h)^2 + k$ have y-intercepts of 2013 and 2014, respectively, and each graph has two positive integer x-intercepts. Find h.

Strategy. The x-intercepts of $y = 3(x-h)^2 + j$ are the roots of $3(x-h)^2 + j = 0$. Use the y-intercept to replace j, divide by 3, and apply Vieta's formulas: the roots have sum $2h$ and a known product. Then do the same for the second graph.

Solution. At $x = 0$ the first graph has $y = 3h^2 + j$, so $3h^2 + j = 2013$. Then

$$3(x-h)^2 + j = 3x^2 - 6hx + 3h^2 + j = 3x^2 - 6hx + 2013,$$

so the x-intercepts of the first graph are the roots of $x^2 - 2hx + 671 = 0$. They are two different positive integers $r < s$ with $rs = 671 = 11 \cdot 61$ and $r + s = 2h$. So $\{r, s\}$ is $\{1, 671\}$ or $\{11, 61\}$, and $h = 336$ or $h = 36$.

In the same way, $2h^2 + k = 2014$, and the x-intercepts of the second graph are the roots of $x^2 - 2hx + 1007 = 0$, with $1007 = 19 \cdot 53$. So $\{r, s\}$ is $\{1, 1007\}$ or $\{19, 53\}$, and $h = 504$ or $h = 36$.

Both conditions hold only for $h = 36$. Then

$$j = 2013 - 3 \cdot 36^2 = -1875 \qquad \text{and} \qquad k = 2014 - 2 \cdot 36^2 = -578,$$

and the x-intercepts are 11 and 61 for the first graph and 19 and 53 for the second.

Answer: 36. □

2. Answer Key

Problem	Letter	Value
1	A	1
2	B	0
3	C	400
4	C	18
5	C	2
6	D	69
7	D	4
8	B	2
9	C	-34
10	B	$-1/9$
11	E	All are true
12	B	6
13	E	80
14		20
15		36