QUADRATICS, THE DISCRIMINANT, AND VIETA'S FORMULAS FOR AMC 10: PRACTICE SET SOLUTIONS

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1. Solutions

Beginner.

1. (2019 AMC 10A, Problem #3) Ana and Bonita were born on the same date in different years, n years apart. Last year Ana was 5 times as old as Bonita. This year Ana's age is the square of Bonita's age. What is n?

(A) 3 **(B)** 5 **(C)** 9 **(D)** 12 **(E)** 15

Strategy. Let a and b be this year's ages. Write one equation for last year and one for this year.

Solution. Let Ana be a and Bonita be b years old this year. Then $a-1=5(b-1)$ and $a=b^2$. Substituting,

$$b^2-1=5b-5 \iff b^2-5b+4=0 \iff (b-1)(b-4)=0.$$

If $b=1$, then $a=1$ and the two were born in the same year, which is not allowed. So $b=4$, $a=16$, and $n=a-b=12$.

Answer: (D) 12. □

2. (1957 AHSME, Problem #36) If $x+y=1$, then the largest value of xy is:

(A) 1 **(B)** 0.5 **(C)** an irrational number about 0.4 **(D)** 0.25 **(E)** 0

Strategy. Use the condition to remove y, so that xy becomes a quadratic in x alone. Its leading coefficient is negative, so its largest value is taken at the vertex.

Solution. Since $y=1-x$,

$$xy=x(1-x)=-x^2+x=-(x^2-x).$$

Completing the square, $x^2-x=\left(x-\frac{1}{2}\right)^2-\frac{1}{4}$, so

$$xy=-\left(x-\frac{1}{2}\right)^2+\frac{1}{4}.$$

The leading coefficient is negative, so by the Maximum and Minimum theorem the largest value of xy is $\frac{1}{4}=0.25$, taken at $x=\frac{1}{2}$, when $y=\frac{1}{2}$ as well.

Answer: (D) 0.25. □

3. (1966 AHSME, Problem #23) If x is real and $4y^2+4xy+x+6=0$, then the complete set of values of x for which y is real, is:

(A) $x\le -2$ or $x\ge 3$ **(B)** $x\le 2$ or $x\ge 3$ **(C)** $x\le -3$ or $x\ge 2$
(D) $-3\le x\le 2$ **(E)** $-2\le x\le 3$

Strategy. For a fixed x, the equation is a quadratic equation in y. A real y exists exactly when its discriminant is at least 0, and this condition is a quadratic inequality in x.

Solution. Fix a real number x. As an equation in y,

$$4y^2+(4x)y+(x+6)=0$$

is a quadratic equation with leading coefficient 4. By the theorem on the number of real roots, it has a real root exactly when its discriminant D is at least 0; the case $D=0$ counts, since a double root is a real root. The numbers -3 and 2 have product -6 and sum -1, so

$$D=(4x)^2-4\cdot 4\cdot(x+6)=16(x^2-x-6)=16(x-3)(x+2).$$

The quadratic $x^2 - x - 6$ has a positive leading coefficient and the roots -2 and 3, so by the Sign of a Quadratic it is negative exactly when $-2 < x < 3$. Hence $D \geq 0$ exactly when $x \leq -2$ or $x \geq 3$. For example, $x = 3$ gives $4y^2 + 12y + 9 = (2y+3)^2 = 0$, so $y = -\frac{3}{2}$, while $x = 0$ gives $4y^2 + 6 = 0$, which has no real root.

Answer: (A) $x \leq -2$ or $x \geq 3$. □

4. (2016 AMC 10B, Problem #9) All three vertices of $\triangle ABC$ lie on the parabola defined by $y = x^2$, with A at the origin and $\overline{BC}$ parallel to the x-axis. The area of the triangle is 64. What is the length of BC?

(A) 4 **(B)** 6 **(C)** 8 **(D)** 10 **(E)** 16

Strategy. Points B and C are on $y = x^2$ at the same height, so by the Symmetry proposition they are symmetric about the axis of symmetry, the line $x = 0$. Write them as $(-t, t^2)$ and (t, t^2) and express the area in terms of t.

Solution. Let $f(x) = x^2$; its vertex is $(0, 0)$, and its axis of symmetry is the line $x = 0$. Write $B = (p, p^2)$ and $C = (q, q^2)$ with $p \neq q$. Since $\overline{BC}$ is parallel to the x-axis, $f(p) = f(q)$, so by part (a) of the Symmetry proposition, $\frac{p+q}{2} = 0$, that is, $q = -p$. As $p \neq q$, $p \neq 0$. Let $t = |p| > 0$. Then B and C are the points $(-t, t^2)$ and (t, t^2) in some order, so $BC = 2t$, and the height of the triangle from $A = (0, 0)$ to the line $y = t^2$ is t^2. The area is

$$\frac{1}{2} \cdot 2t \cdot t^2 = t^3 = 64,$$

so $t = 4$ and $BC = 2t = 8$.

Answer: (C) 8. □

5. (2002 AMC 10B, Problem #10) Suppose that a and b are nonzero real numbers, and that the equation $x^2 + ax + b = 0$ has solutions a and b. What is the pair (a, b)?

(A) $(-2, 1)$ **(B)** $(-1, 2)$ **(C)** $(1, -2)$ **(D)** $(2, -1)$ **(E)** $(4, 4)$

Strategy. The roots of $x^2 + ax + b = 0$ are a and b, so Vieta's formulas give two equations: the sum of the roots, $a + b$, is $-a$, and their product, ab, is b.

Solution. The two roots of $x^2 + ax + b = 0$ are a and b, so by Vieta's formulas

$$a + b = -a \qquad \text{and} \qquad ab = b.$$

Since $b \neq 0$, dividing the second equation by b gives $a = 1$. The first equation then gives $b = -2a = -2$. Check: the equation is $x^2 + x - 2 = 0$, and $x^2 + x - 2 = (x - 1)(x + 2)$, so its roots are 1 and -2, which are a and b.

Answer: (C) $(1, -2)$. □

Intermediate.

6. (2020 AMC 10B, Problem #9) How many ordered pairs of integers (x, y) satisfy the equation

$$x^{2020} + y^2 = 2y?$$

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** infinitely many

Strategy. Move $2y$ to the left side and complete the square in y. The equation then says that two nonnegative integers have sum 1, which leaves only a few cases.

Solution. Moving $2y$ to the left side and completing the square, $y^2 - 2y = (y - 1)^2 - 1$, so the equation is

$$x^{2020} + (y - 1)^2 = 1.$$

Both terms on the left are squares of integers, since $x^{2020} = (x^{1010})^2$, so they are nonnegative integers with sum 1. Hence one of them is 0 and the other is 1.

- If $x^{2020} = 0$, then $x = 0$ and $(y-1)^2 = 1$, so $y - 1 = \pm 1$, that is, $y = 0$ or $y = 2$.
- If $x^{2020} = 1$, then $|x| = 1$, since $x^{2020} \geq 2^{2020} > 1$ when $|x| \geq 2$. So $x = 1$ or $x = -1$, and $(y-1)^2 = 0$ gives $y = 1$.

The solutions are $(0,0)$, $(0,2)$, $(1,1)$, and $(-1,1)$, so there are 4 ordered pairs.

Answer: (D) 4. □

7. (1960 AHSME, Problem #11) For a given value of k, the product of the roots of

$$x^2 - 3kx + 2k^2 - 1 = 0$$

is 7. The roots may be characterized as:

(A) integral and positive **(B)** integral and negative **(C)** rational, but not integral **(D)** irrational **(E)** imaginary

Strategy. By Vieta's formulas the product of the roots is $2k^2 - 1$, and this gives k. Then the discriminant decides whether the roots are real and whether they are rational.

Solution. The discriminant of $x^2 - 3kx + 2k^2 - 1$ is

$$D = (-3k)^2 - 4(2k^2 - 1) = 9k^2 - 8k^2 + 4 = k^2 + 4,$$

which is positive for every k. So the roots are real and different, and by Vieta's formulas their product is $2k^2 - 1$. From $2k^2 - 1 = 7$, $k^2 = 4$, so $k = 2$ or $k = -2$. The equation is then $x^2 - 6x + 7 = 0$ or $x^2 + 6x + 7 = 0$; in both cases the coefficients are integers and $D = k^2 + 4 = 8$. Since 8 is not a perfect square, the roots are irrational by the Rational Roots theorem. Indeed, by the quadratic formula they are $3 \pm \sqrt{2}$ when $k = 2$ and $-3 \pm \sqrt{2}$ when $k = -2$.

Answer: (D) irrational. □

8. (1959 AHSME, Problem #38) If $4x + \sqrt{2x} = 1$, then x:

(A) is an integer **(B)** is fractional **(C)** is irrational **(D)** is imaginary **(E)** may have two different values

Strategy. Let $u = \sqrt{2x}$, so that $u \geq 0$ and $4x = 2u^2$. Then the equation is a quadratic equation in u: solve it by splitting the middle term, and keep only the root with $u \geq 0$.

Solution. Let $u = \sqrt{2x}$. Then $u \geq 0$ and $u^2 = 2x$, so $4x = 2u^2$, and the equation is

$$2u^2 + u - 1 = 0.$$

Here $ac = 2 \cdot (-1) = -2$, and the numbers 2 and -1 have product -2 and sum 1. Splitting the middle term,

$$2u^2 + u - 1 = 2u^2 + 2u - u - 1 = 2u(u+1) - (u+1) = (2u-1)(u+1).$$

By the Zero Product Property, $u = \frac{1}{2}$ or $u = -1$. Only $u = \frac{1}{2}$ satisfies $u \geq 0$, so $2x = u^2 = \frac{1}{4}$ and $x = \frac{1}{8}$. Check: for $x = \frac{1}{8}$, $4x = \frac{1}{2}$ and $\sqrt{2x} = \frac{1}{2}$, and their sum is 1. So x has exactly one value, $\frac{1}{8}$, which is a fraction and not an integer.

Squaring both sides of $1 - 4x = \sqrt{2x}$ instead would give

$$16x^2 - 10x + 1 = (8x-1)(2x-1) = 0,$$

with roots $\frac{1}{8}$ and $\frac{1}{2}$. The value $\frac{1}{2}$ is created by the squaring and does not satisfy the original equation, since $4 \cdot \frac{1}{2} + \sqrt{1} = 3$.

Answer: (B) is fractional. □

9. (2017 AMC 12B, Problem #8) The ratio of the short side of a certain rectangle to the long side is equal to the ratio of the long side to the diagonal. What is the square of the ratio of the short side to the long side of this rectangle?

(A) $\frac{\sqrt{3}-1}{2}$ **(B)** $\frac{1}{2}$ **(C)** $\frac{\sqrt{5}-1}{2}$ **(D)** $\frac{\sqrt{2}}{2}$ **(E)** $\frac{\sqrt{6}-1}{2}$

Strategy. Call the sides $a < b$ and the diagonal d, so that $d^2 = a^2 + b^2$. After squaring, the condition becomes a quadratic equation in $u = \frac{a^2}{b^2}$, the number asked for, and the quadratic formula solves it.

Solution. Let the short side be a, the long side b, and the diagonal d. By the Pythagorean Theorem, $d^2 = a^2 + b^2$. The condition is $\frac{a}{b} = \frac{b}{d}$, and since both ratios are positive, it is equivalent to

$$\frac{a^2}{b^2} = \frac{b^2}{d^2} = \frac{b^2}{a^2 + b^2}.$$

Let $u = \frac{a^2}{b^2}$, the number to find, so $u > 0$. Dividing the numerator and the denominator of the right side by b^2 gives $u = \frac{1}{u+1}$, that is,

$$u^2 + u - 1 = 0.$$

By the quadratic formula,

$$u = \frac{-1 \pm \sqrt{1+4}}{2} = \frac{-1 \pm \sqrt{5}}{2}.$$

Since $\sqrt{5} > 1$, only $\frac{-1+\sqrt{5}}{2}$ is positive, so $u = \frac{\sqrt{5}-1}{2}$. This is less than 1, as it must be, since $a < b$.

Answer: (C) $\frac{\sqrt{5}-1}{2}$. □

10. (1964 AHSME, Problem #30) If

$$(7 + 4\sqrt{3})x^2 + (2 + \sqrt{3})x - 2 = 0,$$

the larger root minus the smaller root is:

(A) $-2 + 3\sqrt{3}$ **(B)** $2 - \sqrt{3}$ **(C)** $6 + 3\sqrt{3}$ **(D)** $6 - 3\sqrt{3}$ **(E)** $3\sqrt{3} + 2$

Strategy. By the Expressions in the Roots proposition, the difference of the roots is $\frac{\sqrt{D}}{|a|}$, so the roots themselves are not needed. The coefficients simplify because $7+4\sqrt{3} = (2+\sqrt{3})^2$.

Solution. Let $m = 2 + \sqrt{3}$. Then $m^2 = 4 + 4\sqrt{3} + 3 = 7 + 4\sqrt{3}$, so the equation is

$$m^2x^2 + mx - 2 = 0.$$

Its discriminant is $D = m^2 - 4 \cdot m^2 \cdot (-2) = 9m^2 > 0$, so the roots r and s are real and different, and $\sqrt{D} = 3m$ because $m > 0$. The leading coefficient m^2 is positive, so by the Expressions in the Roots proposition, the larger root minus the smaller root is

$$|r - s| = \frac{\sqrt{D}}{m^2} = \frac{3m}{m^2} = \frac{3}{m} = \frac{3}{2 + \sqrt{3}}.$$

Since $(2 + \sqrt{3})(2 - \sqrt{3}) = 4 - 3 = 1$, multiplying the numerator and the denominator by $2 - \sqrt{3}$ gives

$$\frac{3}{2 + \sqrt{3}} = \frac{3(2 - \sqrt{3})}{1} = 6 - 3\sqrt{3}.$$

As a check, with $v = mx$ the equation becomes $v^2 + v - 2 = (v + 2)(v - 1) = 0$, so $mx = 1$ or $mx = -2$. The roots are $\frac{1}{m}$ and $-\frac{2}{m}$, and their difference is $\frac{3}{m}$.

Answer: (D) $6 - 3\sqrt{3}$. □

Advanced.

11. (2019 AMC 10A, Problem #19) What is the least possible value of

$$(x + 1)(x + 2)(x + 3)(x + 4) + 2019,$$

where x is a real number?

(A) 2017 **(B)** 2018 **(C)** 2019 **(D)** 2020 **(E)** 2021

Strategy. Multiply the outer two factors and the inner two factors. Both products are $x^2 + 5x$ plus a constant, so the expression is a quadratic in $u = x^2 + 5x + 5$. Find its smallest value, and check that the best value of u occurs for a real x.

Solution. Pairing the factors,

$$(x + 1)(x + 4) = x^2 + 5x + 4 \qquad \text{and} \qquad (x + 2)(x + 3) = x^2 + 5x + 6.$$

Let $u = x^2 + 5x + 5$, so these products are $u - 1$ and $u + 1$, and the expression is

$$(u - 1)(u + 1) + 2019 = u^2 - 1 + 2019 = u^2 + 2018.$$

By the Maximum and Minimum theorem, the quadratic $u^2 + 2018$ has the smallest value 2018, taken only at $u = 0$. This value of u is allowed: the equation $x^2 + 5x + 5 = 0$ has discriminant $25 - 20 = 5 > 0$, so by the theorem on the number of real roots it has two real roots, $x = \frac{-5 \pm \sqrt{5}}{2}$. At these values of x the expression equals 2018, and it is never smaller, so its least possible value is 2018.

Answer: (B) 2018. □

12. (1991 AIME, Problem #1) Find $x^2 + y^2$ if x and y are positive integers such that

$$xy + x + y = 71,$$
$$x^2y + xy^2 = 880.$$

Strategy. Let $s = x + y$ and $p = xy$. The equations give $s + p = 71$ and $sp = 880$, which makes s and p the two roots of one quadratic equation. Then $x^2 + y^2 = s^2 - 2p$.

Solution. Let $s = x + y$ and $p = xy$. Since $x^2y + xy^2 = xy(x + y) = ps$, the equations are

$$s + p = 71 \qquad \text{and} \qquad sp = 880.$$

By the corollary of Vieta's formulas, two numbers with sum 71 and product 880 are the roots of $t^2 - 71t + 880 = 0$. The numbers 16 and 55 have sum 71 and product 880, so $t^2 - 71t + 880 = (t - 16)(t - 55)$, and s and p are 16 and 55 in some order.

If $s = 55$ and $p = 16$, then x and y have sum 55 and product 16, so they are the roots of $t^2 - 55t + 16 = 0$. Its discriminant $55^2 - 4 \cdot 16 = 2961$ is not a perfect square, since $54^2 = 2916$ and $55^2 = 3025$, so by the Rational Roots theorem these roots are irrational, and x and y would not be integers. Hence $s = 16$ and $p = 55$. Then x and y are the roots of $t^2 - 16t + 55 = (t - 5)(t - 11) = 0$, that is, 5 and 11 in some order. Indeed $5 \cdot 11 + 5 + 11 = 71$ and $5 \cdot 11 \cdot 16 = 880$. Finally, by the first identity of the Expressions in the Roots proposition,

$$x^2 + y^2 = s^2 - 2p = 256 - 110 = 146.$$

Answer: 146. □

13. (2015 AMC 10A, Problem #23) The zeros of the function

$$f(x) = x^2 - ax + 2a$$

are integers. What is the sum of the possible values of a?

(A) 7 **(B)** 8 **(C)** 16 **(D)** 17 **(E)** 18

Strategy. If the zeros are integers, then their sum a is an integer and the discriminant $a^2 - 8a$ is a perfect square. Completing the square turns this into a difference of two squares equal to 16, which can be factored in only a few ways.

Solution. Let the zeros be the integers r and s, with $r = s$ if f has a double root. By Vieta's formulas, $a = r + s$, so a is an integer, and f has integer coefficients. Its discriminant is $D = a^2 - 8a$, and since the roots are integers, D is a perfect square by the Rational Roots theorem: $D = m^2$ for an integer $m \geq 0$. Completing the square, $a^2 - 8a = (a-4)^2 - 16$, so

$$(a-4)^2 - m^2 = 16, \qquad \text{that is,} \qquad (a-4-m)(a-4+m) = 16.$$

The two factors differ by $2m$, so both are even or both are odd, and since their product is even, both are even. Also $a - 4 - m \leq a - 4 + m$. So the two factors, in order, are 2 and 8, 4 and 4, -8 and -2, or -4 and -4. Their sum is $2(a-4)$, which is 10, 8, -10, or -8, so a is 9, 8, -1, or 0. Each of these values works: for $a = 9$, 8, -1, and 0, the function is $(x-3)(x-6)$, $(x-4)^2$, $(x-1)(x+2)$, and x^2, and all its zeros are integers. The sum of the possible values of a is $9 + 8 + (-1) + 0 = 16$.

Answer: (C) 16. □

14. (1985 AHSME, Problem #19) Consider the graphs of $y = Ax^2$ and $y^2 + 3 = x^2 + 4y$, where A is a positive constant and x and y are real variables. In how many points do the two graphs intersect?

(A) exactly 4 **(B)** exactly 2 **(C)** at least 1, but the number varies for different positive values of A **(D)** 0 for at least one positive value of A **(E)** none of these

Strategy. On the first graph, $x^2 = \frac{y}{A}$. Substituting this into the second equation gives a quadratic equation in y. Its discriminant and Vieta's formulas show that it has two different positive roots, and each positive value of y gives two values of x.

Solution. A point (x, y) is on both graphs exactly when $y = Ax^2$ and $y^2 + 3 = x^2 + 4y$. The first equation says that $x^2 = \frac{y}{A}$, so the second becomes

$$y^2 + 3 = \frac{y}{A} + 4y, \qquad \text{that is,} \qquad y^2 - \left(4 + \frac{1}{A}\right) y + 3 = 0.$$

Since $A > 0$, $4 + \frac{1}{A} > 4$, so the discriminant is

$$D = \left(4 + \frac{1}{A}\right)^2 - 12 > 16 - 12 > 0.$$

By the theorem on the number of real roots, this equation has two different real roots y_1 and y_2. By Vieta's formulas, $y_1 y_2 = 3 > 0$, so the two roots have the same sign, and $y_1 + y_2 = 4 + \frac{1}{A} > 0$, so both are positive. For each root y_i, the equation $x^2 = \frac{y_i}{A}$ has the two solutions $x = \pm\sqrt{y_i/A}$, and each point (x, y_i) satisfies both original equations. So the two graphs meet in exactly these four different points, for every positive A.

Answer: (A) exactly 4. □

15. (2010 AMC 12B, Problem #23) Monic quadratic polynomials $P(x)$ and $Q(x)$ have the property that $P(Q(x))$ has zeros at $x = -23$, -21, -17, and -15, and $Q(P(x))$ has zeros at $x = -59$, -57, -51, and -49. What is the sum of the minimum values of $P(x)$ and $Q(x)$?

(A) -100 **(B)** -82 **(C)** -73 **(D)** -64 **(E)** 0

Strategy. The number $P(Q(x))$ is 0 exactly when $Q(x)$ is a root of P, and two different inputs at which Q takes the same value are symmetric about the axis of symmetry of Q. So the four zeros form two pairs with the same average, which locates the axis. Do the same for P, and then apply Vieta's formulas to the roots of P and of Q.

Solution. "Monic" means that the leading coefficient is 1. By the Vertex Form proposition, $P(x) = (x-p)^2 + m$ and $Q(x) = (x-q)^2 + n$ for some numbers p, m, q, and n. The axes of symmetry are the lines $x = p$ and $x = q$, and by the Maximum and Minimum theorem the minimum values of P and Q are m and n.

The number $P(Q(x))$ is 0 exactly when $Q(x)$ is a root of P. So the four numbers $Q(-23)$, $Q(-21)$, $Q(-17)$, and $Q(-15)$ are roots of P, and they take at most two different values. For each number c, the equation $Q(x) = c$ is a quadratic equation, so it has at most two solutions. Hence Q takes two different values, each at exactly two of the four zeros. By part (a) of the Symmetry proposition, two different inputs at which Q takes the same value have average q, so the four zeros split into two pairs with the same average. The three ways to split -23, -21, -17, -15 into two pairs give the averages -22 and -16, -20 and -18, or -19 and -19. Only the last split, $\{-23, -15\}$ and $\{-21, -17\}$, has equal averages, so $q = -19$. In the same way, the only split of -59, -57, -51, -49 into two pairs with equal averages is $\{-59, -49\}$ and $\{-57, -51\}$, so $p = -54$.

Now $Q(-23) = 16 + n$ and $Q(-21) = 4 + n$ are the two different values of Q at the zeros, so they are the two roots of P. Since $P(x) = (x+54)^2 + m = x^2 + 108x + 2916 + m$, Vieta's formulas give

$$(16 + n) + (4 + n) = -108, \qquad \text{so} \qquad n = -64.$$

Similarly, the roots of Q are $P(-59) = 25 + m$ and $P(-57) = 9 + m$. Since $Q(x) = (x+19)^2 + n = x^2 + 38x + 361 + n$, Vieta's formulas give

$$(25 + m) + (9 + m) = -38, \qquad \text{so} \qquad m = -36.$$

The sum of the minimum values is $m + n = -36 + (-64) = -100$.

Check: $P(x) = (x+54)^2 - 36$ has the roots -48 and -60, and $Q(x) = (x+19)^2 - 64$ has the roots -11 and -27. The equation $Q(x) = -48$ gives $(x+19)^2 = 16$, so $x = -15$ or $x = -23$, and $Q(x) = -60$ gives $(x+19)^2 = 4$, so $x = -17$ or $x = -21$. Likewise $P(x) = -11$ gives $x = -49$ or $x = -59$, and $P(x) = -27$ gives $x = -51$ or $x = -57$.

Answer: (A) -100. □

2. Answer Key

Problem	Letter	Value
1	D	12
2	D	0.25
3	A	$x \leq -2$ or $x \geq 3$
4	C	8
5	C	$(1, -2)$
6	D	4
7	D	irrational
8	B	is fractional
9	C	$(\sqrt{5}-1)/2$
10	D	$6 - 3\sqrt{3}$
11	B	2018
12		146
13	C	16
14	A	exactly 4
15	A	-100