

SYSTEMS, RATES, AND WORK PROBLEMS FOR AMC 10

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CONTENTS

1. Linear Systems	2
1.1. Substitution and Elimination	2
1.2. Three-Variable Systems	3
1.3. Adding All the Equations	3
2. Distance, Rate, and Time	5
2.1. Average Speed	5
2.2. Relative Speed	6
3. Combined Work	7
3.1. Work Rates	7
3.2. Filling and Draining	8
4. Mixtures	9
4.1. Concentration	9
4.2. Weighted Averages	10
5. Problems	12

1. LINEAR SYSTEMS

A word problem often becomes a few linear equations in a few unknowns. This section covers how to solve such a system and how to find the quantity a problem asks for without solving for every unknown.

1.1. Substitution and Elimination.

Definition 1.1.1. A **linear equation** in the unknowns $x, y, z, \dots$ is an equation of the form $ax + by + cz + \dots = d$, where $a, b, c, \dots, d$ are constants. A **system** is a set of equations in the same unknowns. A **solution** of a system is a choice of values for the unknowns that satisfies every equation at once.

Two methods solve any linear system.

- **Substitution.** Solve one equation for one unknown and substitute the result into the other equations. This removes one unknown and one equation.
- **Elimination.** Multiply equations by constants, then add or subtract them so that one unknown cancels.

Substitution is quicker when an equation already gives one unknown in terms of the others, as in “the second is seven times the third.” Elimination is quicker when the coefficients of one unknown already match, or match after one equation is multiplied by a constant.

Remark 1.1.2. Finding n unknowns usually takes n equations. Without extra conditions (such as whole-number or positive unknowns), fewer equations than unknowns cannot determine every unknown, but they may still determine a combination of them, such as a sum or an average. In that case, look for multiples of the given equations that add up to the combination asked for.

Example.

Problem 1.1.3 (2009 AMC 10B, Problem #15). When a bucket is two-thirds full of water, the bucket and water weigh a kilograms. When the bucket is one-half full of water the total weight is b kilograms. In terms of a and b , what is the total weight in kilograms when the bucket is full of water?

- (A) $\frac{2}{3}a + \frac{1}{3}b$ (B) $\frac{3}{2}a - \frac{1}{2}b$ (C) $\frac{3}{2}a + b$ (D) $\frac{3}{2}a + 2b$ (E) $3a - 2b$

Strategy. Let x be the weight of the empty bucket and y the weight of the water in a full bucket. Each sentence gives a linear equation in x and y , and the question asks for $x + y$.

Solution. Two-thirds of the water weighs $\frac{2}{3}y$ and half of it weighs $\frac{1}{2}y$, so

$$x + \frac{2}{3}y = a, \quad x + \frac{1}{2}y = b.$$

Subtracting the second equation from the first eliminates x and gives $\frac{1}{6}y = a - b$, so $y = 6a - 6b$. Substituting this into the second equation gives $x = b - \frac{1}{2}(6a - 6b) = 4b - 3a$, so the full bucket weighs $x + y = (4b - 3a) + (6a - 6b) = 3a - 2b$.

The method of the remark finds $x + y$ without finding x and y . Look for numbers m and n with $m(x + \frac{2}{3}y) + n(x + \frac{1}{2}y) = x + y$. Matching the coefficients of x and y gives $m + n = 1$ and $\frac{2}{3}m + \frac{1}{2}n = 1$, so $m = 3$ and $n = -2$. Then $x + y = 3a - 2b$ directly.

Answer: (E) $3a - 2b$. □

1.2. Three-Variable Systems. To determine three unknowns uniquely, a linear system needs three independent equations: each must give information not implied by the others. Use one pair of equations to eliminate an unknown, then a different pair to eliminate the same unknown. Solve the resulting two-variable system, then substitute back to find the third unknown. If elimination gives a contradiction such as $0 = 1$, there is no solution; if an equation reduces to $0 = 0$, it adds no restriction.

Word problems often state the equations as relations, such as “the first is 4 times the sum of the other two.” Then substitution is quicker: write every unknown in terms of one of them. If an expression such as $b + c$ appears in several equations, treat it as a single unknown.

Examples.

Problem 1.2.1. Solve the system

$$\begin{aligned}x + y + z &= 6, \\2x - y + z &= 3, \\x + 2y - z &= 2.\end{aligned}$$

Strategy. The coefficients of z are 1, 1, and -1 , so adding the third equation to each of the first two eliminates z .

Solution. Adding the first and third equations gives $2x + 3y = 8$. Adding the second and third equations gives $3x + y = 5$, so $y = 5 - 3x$. Substituting this into $2x + 3y = 8$ gives $2x + 15 - 9x = 8$, so $7x = 7$ and $x = 1$. Then $y = 5 - 3 = 2$, and the first equation gives $z = 6 - 1 - 2 = 3$. The other two equations check: $2 - 2 + 3 = 3$ and $1 + 4 - 3 = 2$.

Answer: $(x, y, z) = (1, 2, 3)$. □

Problem 1.2.2 (2003 AMC 10A, Problem #13). The sum of three numbers is 20. The first is 4 times the sum of the other two. The second is seven times the third. What is the product of all three?

(A) 28 (B) 40 (C) 100 (D) 400 (E) 800

Strategy. Call the numbers a , b , and c . The first two conditions both involve $b + c$, so treat $b + c$ as one unknown.

Solution. The three conditions are

$$a + b + c = 20, \quad a = 4(b + c), \quad b = 7c.$$

The first equation gives $b + c = 20 - a$. Substituting this into the second, $a = 4(20 - a)$, so $5a = 80$ and $a = 16$. Then $b + c = 4$, and with $b = 7c$ this gives $8c = 4$, so $c = \frac{1}{2}$ and $b = \frac{7}{2}$. The product is

$$16 \cdot \frac{7}{2} \cdot \frac{1}{2} = 28.$$

Answer: (A) 28. □

1.3. Adding All the Equations. When each equation gives the sum of some of the unknowns and every unknown appears in the same number of equations, add all the equations at once. The sum is a multiple of the total of the unknowns, so dividing gives the total. If each equation leaves out exactly one unknown, subtracting one of the equations from the total leaves the unknown that equation is missing.

Proposition 1.3.1. If $x + y = p$, $y + z = q$, and $z + x = r$, then

$$x + y + z = \frac{p + q + r}{2},$$

so $x = \frac{p+q+r}{2} - q$, $y = \frac{p+q+r}{2} - r$, and $z = \frac{p+q+r}{2} - p$.

Proof. Each unknown appears in exactly two of the three equations, so adding them gives $2(x + y + z) = p + q + r$. Subtracting $y + z = q$ from $x + y + z$ leaves x , and the same works for y and z . $\square$

The same method works for four unknowns whose sums three at a time are known. Each unknown appears in three of the four equations, so the sum of the equations is three times the total.

Two other setups lead to the same idea.

- When a problem asks for a combination such as $A + B + C$ or an average, look for constants to multiply the equations by so that their sum is exactly that combination.
- When several expressions are all equal, as in $a + 1 = b + 2 = c + 3$, call the common value k . Then each unknown is k minus a constant, and adding them gives their total in terms of k .

Examples.

Problem 1.3.2 (2012 AMC 10A, Problem #8). The sums of three whole numbers taken in pairs are 12, 17, and 19. What is the middle number?

(A) 4 (B) 5 (C) 6 (D) 7 (E) 8

Strategy. Match the three sums to the three pairs, then add all three equations.

Solution. Let the numbers be $a \leq b \leq c$. Then $a + b \leq a + c \leq b + c$, so

$$a + b = 12, \quad a + c = 17, \quad b + c = 19.$$

Adding the three equations gives $2(a + b + c) = 48$, so $a + b + c = 24$. The middle number is the one missing from $a + c$:

$$b = (a + b + c) - (a + c) = 24 - 17 = 7.$$

The other two numbers are $a = 24 - 19 = 5$ and $c = 24 - 12 = 12$, and indeed $5 + 7 = 12$, $5 + 12 = 17$, and $7 + 12 = 19$.

Answer: (D) 7. $\square$

Problem 1.3.3 (2002 AMC 10A, Problem #9). Suppose A , B , and C are three numbers for which $1001C - 2002A = 4004$ and $1001B + 3003A = 5005$. The average of the three numbers A , B , and C is

(A) 1 (B) 3 (C) 6 (D) 9 (E) not uniquely determined

Strategy. Two equations cannot determine three unknowns, but the question asks only for $\frac{A+B+C}{3}$. Simplify the equations and look for a combination of them equal to $A + B + C$.

Solution. Dividing both equations by 1001 gives

$$C - 2A = 4, \quad B + 3A = 5.$$

Adding them, the terms $-2A$ and $3A$ combine to A :

$$A + B + C = 9.$$

So the average is $\frac{9}{3} = 3$. The numbers themselves are not determined: $(A, B, C) = (0, 5, 4)$ and $(A, B, C) = (1, 2, 6)$ both satisfy the equations, and both have average 3.

Answer: (B) 3. $\square$

2. DISTANCE, RATE, AND TIME

An object that moves at a constant rate r for a time t covers the distance

$$d = rt.$$

The units must agree. With a rate in miles per hour, the time must be in hours, so convert minutes first (3 minutes is $\frac{1}{20}$ hour). When a problem has several people or several parts of a trip, write one equation $d = rt$ for each and look for what they share: the same distance, the same time, or a known difference. A table with one row for each part and columns for rate, time, and distance keeps the equations organized.

2.1. Average Speed.

Definition 2.1.1. The **average speed** for a trip is the total distance divided by the total time.

Average speed is not the average of the speeds. Over two equal distances, a trip spends more time at the lower speed, so the lower speed counts for more.

Proposition 2.1.2. If a trip covers the same distance twice, once at speed u and once at speed v , its average speed is

$$\frac{2uv}{u+v}.$$

Proof. Let each part have length d . The two parts take $\frac{d}{u}$ and $\frac{d}{v}$ units of time, so the average speed is

$$\frac{2d}{\frac{d}{u} + \frac{d}{v}} = \frac{2}{\frac{1}{u} + \frac{1}{v}} = \frac{2uv}{u+v}.$$

□

Examples.

Problem 2.1.3 (2003 AMC 10A, Problem #4). It takes Mary 30 minutes to walk uphill 1 km from her home to school, but it takes her only 10 minutes to walk from school to home along the same route. What is her average speed, in km/hr, for the round trip?

(A) 3 (B) 3.125 (C) 3.5 (D) 4 (E) 4.5

Strategy. Divide the total distance by the total time. Do not average the two speeds.

Solution. The round trip is 2 km long and takes $30 + 10 = 40$ minutes, which is $\frac{2}{3}$ hour. Her average speed is

$$\frac{2}{2/3} = 3 \text{ km/hr.}$$

The proposition gives the same result: Mary walks uphill at 2 km/hr and downhill at 6 km/hr, and $\frac{2 \cdot 2 \cdot 6}{2+6} = 3$. The average of the two speeds is 4, which is not her average speed.

Answer: (A) 3. □

Problem 2.1.4 (2002 AMC 10A, Problem #12). Mr. Earl E. Bird leaves his house for work at exactly 8:00 A.M. every morning. When he averages 40 miles per hour, he arrives at his workplace three minutes late. When he averages 60 miles per hour, he arrives three minutes early. At what average speed, in miles per hour, should Mr. Bird drive to arrive at his workplace precisely on time?

(A) 45 (B) 48 (C) 50 (D) 55 (E) 58

Strategy. Let d be the distance in miles and t the number of hours the drive should take. Both trips cover the same distance, so their two expressions for d are equal.

Solution. Three minutes is $\frac{1}{20}$ hour, so the two trips are:

Trip	Rate	Time	Distance
Late	40	$t + \frac{1}{20}$	$40(t + \frac{1}{20})$
Early	60	$t - \frac{1}{20}$	$60(t - \frac{1}{20})$

Both distances equal d , so

$$40t + 2 = 60t - 3, \quad \text{so} \quad t = \frac{1}{4}.$$

Then $d = 40(\frac{1}{4} + \frac{1}{20}) = 40 \cdot \frac{3}{10} = 12$ miles. To cover 12 miles in $\frac{1}{4}$ hour, he must average $12 \div \frac{1}{4} = 48$ miles per hour.

The answer is not the average of 40 and 60. The on-time travel time t is the average of the two travel times $\frac{d}{40}$ and $\frac{d}{60}$, so the required speed is

$$\frac{d}{t} = \frac{2d}{\frac{d}{40} + \frac{d}{60}} = \frac{2 \cdot 40 \cdot 60}{40 + 60} = 48,$$

as in the proof of the proposition with $u = 40$ and $v = 60$.

Answer: (B) 48. □

2.2. Relative Speed. When two objects move along the same line, it is often easier to follow the distance between them than their positions.

- If they move toward each other, the distance between them shrinks at the sum of their speeds.
- If they move in the same direction, the faster one gains on the slower one at the difference of their speeds.

The same rule applies when the ground itself moves. A boat whose speed in still water is b , on a river whose current has speed c , moves downstream at $b + c$ and upstream at $b - c$. A person walking down a moving escalator moves at the walking speed plus the speed of the escalator.

Examples.

Problem 2.2.1 (2006 AMC 12B, Problem #5). John is walking east at a speed of 3 miles per hour, while Bob is also walking east, but at a speed of 5 miles per hour. If Bob is now 1 mile west of John, how many minutes will it take for Bob to catch up to John?

(A) 30 (B) 50 (C) 60 (D) 90 (E) 120

Strategy. They walk in the same direction, so the gap between them closes at the difference of their speeds.

Solution. Bob gains $5 - 3 = 2$ miles per hour on John. The gap is 1 mile, so it closes in $\frac{1}{2}$ hour, which is 30 minutes.

Answer: (A) 30. □

Problem 2.2.2 (2021 Fall AMC 10A, Problem #11). Emily sees a ship traveling at a constant speed along a straight section of a river. She walks parallel to the riverbank at a uniform rate faster than the ship. She counts 210 equal steps walking from the back of the ship to the front. Walking in the opposite direction, she counts 42 steps of the same size from the front of the ship to the back. In terms of Emily's equal steps, what is the length of the ship?

(A) 70 (B) 84 (C) 98 (D) 105 (E) 126

Strategy. Measure lengths in steps and give Emily speed 1. Walking with the ship, she gains on it at the difference of the speeds; walking against it, the gap closes at the sum. Each walk covers the length of the ship relative to the ship.

Solution. Measure distance in Emily's steps and time in the time she takes for one step. Then Emily's speed is 1, and the ship's speed is some number k with $0 < k < 1$. Let L be the length of the ship.

Walking from the back to the front, Emily moves in the same direction as the ship, so she gains on the ship at the rate $1 - k$. She takes 210 steps, so 210 units of time pass, and in that time she gains the full length of the ship:

$$L = 210(1 - k).$$

Walking from the front to the back, she moves against the ship, so the distance closes at the rate $1 + k$, and

$$L = 42(1 + k).$$

Setting the two expressions equal gives $210 - 210k = 42 + 42k$, so $252k = 168$ and $k = \frac{2}{3}$. Then $L = 210 \cdot \frac{1}{3} = 70$.

Answer: (A) 70. □

3. COMBINED WORK

Work problems are rate problems in which the distance is replaced by the amount of a job that has been done.

3.1. Work Rates.

Definition 3.1.1. If a worker can do a job alone in t hours, the worker's **rate** is $\frac{1}{t}$ of the job per hour. The amount of work done is

$$\text{work} = \text{rate} \times \text{time},$$

in the same way that $d = rt$.

Proposition 3.1.2. When workers work together, their rates add. If one worker can do a job in a hours and another in b hours, then together they do $\frac{1}{a} + \frac{1}{b}$ of the job per hour and finish it in

$$\frac{1}{\frac{1}{a} + \frac{1}{b}} = \frac{ab}{a + b}$$

hours.

Add rates, not times. Two workers who each need 6 hours finish together in 3 hours. When a job is measured in units, such as bricks or cupcakes, measure the rates in units per hour instead of jobs per hour.

Examples.

Problem 3.1.3 (2008 AMC 10A, Problem #13). Doug can paint a room in 5 hours. Dave can paint the same room in 7 hours. Doug and Dave paint the room together and take a one-hour break for lunch. Let t be the total time, in hours, required for them to complete the job working together, including lunch. Which of the following equations is satisfied by t ?

- (A) $\left(\frac{1}{5} + \frac{1}{7}\right)(t + 1) = 1$ (B) $\left(\frac{1}{5} + \frac{1}{7}\right)t + 1 = 1$ (C) $\left(\frac{1}{5} + \frac{1}{7}\right)t = 1$
 (D) $\left(\frac{1}{5} + \frac{1}{7}\right)(t - 1) = 1$ (E) $(5 + 7)t = 1$

Strategy. Convert each time to a rate and add the rates. Work is done only while they paint, which is $t - 1$ hours.

Solution. Doug paints $\frac{1}{5}$ of the room per hour and Dave paints $\frac{1}{7}$, so together they paint $\frac{1}{5} + \frac{1}{7}$ of the room per hour. They paint for $t - 1$ hours, since the lunch hour is part of t , and in that time they paint one room. Since work equals rate times time,

$$\left(\frac{1}{5} + \frac{1}{7}\right)(t - 1) = 1.$$

(Solving gives $t - 1 = \frac{35}{12}$, so $t = \frac{47}{12}$ hours.)

Answer: (D) $\left(\frac{1}{5} + \frac{1}{7}\right)(t - 1) = 1.$ □

Problem 3.1.4 (2008 AMC 10B, Problem #18). Bricklayer Brenda would take 9 hours to build a chimney alone, and bricklayer Brandon would take 10 hours to build it alone. When they work together, they talk a lot, and their combined output is decreased by 10 bricks per hour. Working together, they build the chimney in 5 hours. How many bricks are in the chimney?

- (A) 500 (B) 900 (C) 950 (D) 1000 (E) 1900

Strategy. The slowdown is given in bricks per hour, so measure the rates in bricks per hour. If the chimney has N bricks, Brenda lays $\frac{N}{9}$ bricks per hour.

Solution. Let the chimney have N bricks. Brenda lays $\frac{N}{9}$ bricks per hour and Brandon lays $\frac{N}{10}$. Working together they lay $\frac{N}{9} + \frac{N}{10} - 10$ bricks per hour, and in 5 hours they lay all N bricks:

$$5\left(\frac{N}{9} + \frac{N}{10} - 10\right) = N.$$

This is $\frac{5N}{9} + \frac{N}{2} - 50 = N$. Multiplying by 18 gives $10N + 9N - 900 = 18N$, so $N = 900$. Check: together they lay $100 + 90 - 10 = 180$ bricks per hour, and $5 \cdot 180 = 900$.

Answer: (B) 900. □

3.2. Filling and Draining. A pipe that fills a tank in t hours is a worker with rate $\frac{1}{t}$ of the tank per hour. A drain or a leak works against the job, so its rate is subtracted: if water flows into a tank at 10 gallons per minute and out at 8 gallons per minute, the amount of water in the tank grows at 2 gallons per minute. When a problem gives the time for several groups of pipes, each group gives one linear equation in the unknown rates, and the methods of Section 1 apply.

Example.

Problem 3.2.1 (1973 AHSME, Problem #14). Each valve A , B , and C , when open, releases water into a tank at its own constant rate. With all three valves open, the tank fills in 1 hour, with only valves A and C open it takes 1.5 hours, and with only valves B and C open it takes 2 hours. The number of hours required with only valves A and B open is

- (A) 1.1 (B) 1.15 (C) 1.2 (D) 1.25 (E) 1.75

Strategy. Let a , b , and c be the rates in tanks per hour. Each sentence gives one linear equation, and the question asks for $a + b$.

Solution. A group of valves that fills the tank in t hours has total rate $\frac{1}{t}$, so

$$a + b + c = 1, \quad a + c = \frac{1}{1.5} = \frac{2}{3}, \quad b + c = \frac{1}{2}.$$

Subtracting the second equation from the first gives $b = \frac{1}{3}$, and subtracting the third from the first gives $a = \frac{1}{2}$. So valves A and B together fill $\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$ of the tank per hour, and they need $\frac{6}{5} = 1.2$ hours.

Answer: (C) 1.2. □

4. MIXTURES

A mixture problem combines ingredients, such as acid and water, and asks about the percent of one ingredient. Track the amount of each ingredient rather than the percents: when mixtures are combined, the amounts add and the percents do not.

4.1. Concentration.

Definition 4.1.1. The **concentration** of an ingredient in a mixture is the fraction (or percent) of the mixture that is that ingredient. A mixture of total amount V with concentration p contains pV of the ingredient.

A well-stirred mixture has the same concentration throughout, so any part of it contains each ingredient in the same proportion as the whole. Three more facts follow from the definition.

- Adding water changes the total but not the amount of the ingredient.
- Adding the pure ingredient increases the amount of the ingredient and the total by the same quantity.
- When two mixtures are combined, the amounts of the ingredient add and the totals add.

For each stage of a problem, write one equation: amount of ingredient = concentration $\times$ total.

Examples.

Problem 4.1.2 (1953 AHSME, Problem #9). The number of ounces of water needed to reduce 9 ounces of shaving lotion containing 50% alcohol to a lotion containing 30% alcohol is:

- (A) 3 (B) 4 (C) 5 (D) 6 (E) 7

Strategy. Adding water does not change the amount of alcohol, so track the alcohol.

Solution. The lotion contains $0.5 \cdot 9 = 4.5$ ounces of alcohol. After x ounces of water are added, there are $9 + x$ ounces of lotion with the same 4.5 ounces of alcohol, and this must be 30% of the lotion:

$$4.5 = 0.3(9 + x), \quad \text{so} \quad 9 + x = 15 \quad \text{and} \quad x = 6.$$

Answer: (D) 6. □

Problem 4.1.3 (1973 AHSME, Problem #33). When one ounce of water is added to a mixture of acid and water, the new mixture is 20% acid. When one ounce of acid is added to the new mixture, the result is $33\frac{1}{3}\%$ acid. The percentage of acid in the original mixture is

(A) 22% (B) 24% (C) 25% (D) 30% (E) $33\frac{1}{3}\%$

Strategy. Let the original mixture contain a ounces of acid and w ounces of water. Track both through the two steps. Each percent gives one linear equation in a and w .

Solution. After one ounce of water is added, the mixture has a ounces of acid out of $a + w + 1$ ounces, and it is 20% acid:

$$a = \frac{1}{5}(a + w + 1), \quad \text{so} \quad w = 4a - 1.$$

After one ounce of acid is added, it has $a + 1$ ounces of acid out of $a + w + 2$ ounces, and it is $33\frac{1}{3}\%$, or $\frac{1}{3}$, acid:

$$a + 1 = \frac{1}{3}(a + w + 2), \quad \text{so} \quad w = 2a + 1.$$

Then $4a - 1 = 2a + 1$, so $a = 1$ and $w = 3$. The original mixture is $\frac{1}{1+3} = 25\%$ acid. Check: adding water gives $\frac{1}{5} = 20\%$ acid, and then adding acid gives $\frac{2}{6} = 33\frac{1}{3}\%$ acid.

Answer: (C) 25%. □

4.2. Weighted Averages. When two mixtures are combined, the concentration of the result is a **weighted average** of their concentrations: each concentration counts in proportion to the amount of its mixture.

Proposition 4.2.1. Mix positive amounts V_1 and V_2 with concentrations p_1 and p_2 , respectively, where $p_1 < p_2$. The result has concentration

$$p = \frac{p_1 V_1 + p_2 V_2}{V_1 + V_2}.$$

Moreover, $p_1 < p < p_2$, and

$$V_1(p - p_1) = V_2(p_2 - p).$$

Proof. The amounts of the ingredient add and the totals add, which gives the formula for p . Multiplying it by $V_1 + V_2$ and rearranging gives $V_1(p - p_1) = V_2(p_2 - p)$. Since V_1 and V_2 are positive, $p - p_1$ and $p_2 - p$ have the same sign. Their sum $p_2 - p_1$ is positive, so both are positive. □

The last equation is a balance: $(p - p_1) : (p_2 - p) = V_2 : V_1$, so the larger amount pulls p toward its own concentration. The same rule applies to any average over two groups, such as an average score over two classes or a tax rate that is different on two parts of an income.

Examples.

Problem 4.2.2. How many liters of a 20% acid solution must be added to 30 liters of a 50% acid solution to make a 30% acid solution?

Strategy. The amounts of acid add and the volumes add. The balance $V_1(p - p_1) = V_2(p_2 - p)$ gives x directly.

Solution. Let x be the number of liters of the 20% solution. The acid in the two solutions adds up to the acid in the mixture:

$$0.2x + 0.5 \cdot 30 = 0.3(x + 30), \quad \text{so} \quad 0.2x + 15 = 0.3x + 9 \quad \text{and} \quad x = 60.$$

With the balance, $x(30 - 20) = 30(50 - 30)$, so $10x = 600$ and $x = 60$ again.

Answer: 60. □

Problem 4.2.3 (2001 AMC 10, Problem #9). The state income tax where Kristin lives is levied at the rate of $p\%$ of the first \$28000 of annual income plus $(p+2)\%$ of any amount above \$28000. Kristin noticed that the state income tax she paid amounted to $(p+0.25)\%$ of her annual income. What was her annual income?

- (A) \$28000 (B) \$32000 (C) \$35000 (D) \$42000 (E) \$56000

Strategy. The overall rate $(p+0.25)\%$ is a weighted average of the rates $p\%$ and $(p+2)\%$ on the two parts of the income, weighted by the sizes of the parts.

Solution. An income of at most \$28000 is taxed at exactly $p\%$, so Kristin's income is $28000 + x$ dollars for some $x > 0$. The overall rate is the weighted average of $p\%$ on 28000 dollars and $(p+2)\%$ on x dollars. These rates are 0.25 and 1.75 percentage points from the overall rate, so the balance in the proposition gives $28000 \cdot 0.25 = 1.75x$, and $x = 4000$. Her income is $28000 + 4000 = 32000$ dollars.

Answer: (B) \$32000.

□

5. PROBLEMS

Beginner.

1. (2012 AMC 10A, Problem #1) Cagney can frost a cupcake every 20 seconds and Lacey can frost a cupcake every 30 seconds. Working together, how many cupcakes can they frost in 5 minutes?
(A) 10 (B) 15 (C) 20 (D) 25 (E) 30
2. (2018 AMC 10B, Problem #2) Sam drove 96 miles in 90 minutes. His average speed during the first 30 minutes was 60 mph (miles per hour), and his average speed during the second 30 minutes was 65 mph. What was his average speed, in mph, during the last 30 minutes?
(A) 64 (B) 65 (C) 66 (D) 67 (E) 68
3. (2006 AMC 10B, Problem #9) Francesca uses 100 grams of lemon juice, 100 grams of sugar, and 400 grams of water to make lemonade. There are 25 calories in 100 grams of lemon juice and 386 calories in 100 grams of sugar. Water contains no calories. How many calories are in 200 grams of her lemonade?
(A) 129 (B) 137 (C) 174 (D) 223 (E) 411
4. (1990 AHSME, Problem #15) Four whole numbers, when added three at a time, give the sums 180, 197, 208, and 222. What is the largest of the four numbers?
(A) 77 (B) 83 (C) 89 (D) 95 (E) cannot be determined from the given information
5. (2008 AMC 10A, Problem #11) While Steve and LeRoy are fishing 1 mile from shore, their boat springs a leak, and water comes in at a constant rate of 10 gallons per minute. The boat will sink if it takes in more than 30 gallons of water. Steve starts rowing toward the shore at a constant rate of 4 miles per hour while LeRoy bails water out of the boat. What is the slowest rate, in gallons per minute, at which LeRoy can bail if they are to reach the shore without sinking?
(A) 2 (B) 4 (C) 6 (D) 8 (E) 10

Intermediate.

6. (2012 AMC 10B, Problem #13) It takes Clea 60 seconds to walk down an escalator when it is not operating, and only 24 seconds to walk down the escalator when it is operating. How many seconds does it take Clea to ride down the operating escalator when she just stands on it?
(A) 36 (B) 40 (C) 42 (D) 48 (E) 52
7. (2002 AMC 10A, Problem #16) If $a + 1 = b + 2 = c + 3 = d + 4 = a + b + c + d + 5$, then $a + b + c + d$ is
(A) -5 (B) $-\frac{10}{3}$ (C) $-\frac{7}{3}$ (D) $\frac{5}{3}$ (E) 5

8. (1986 AHSME, Problem #12) John scores 93 on this year's AHSME. Had the old scoring system still been in effect, he would score only 84 for the same answers. How many questions does he leave unanswered? (In the new scoring system that year, one received 5 points for each correct answer, 0 points for each wrong answer, and 2 points for each problem left unanswered. In the previous scoring system, one started with 30 points, received 4 more for each correct answer, lost 1 point for each wrong answer, and neither gained nor lost points for unanswered questions. There are 30 questions in the 1986 AHSME.)
(A) 6 (B) 9 (C) 11 (D) 14 (E) not uniquely determined
9. (1967 AHSME, Problem #8) To m ounces of an $m\%$ solution of acid, x ounces of water are added to yield an $(m - 10)\%$ solution. If $m > 25$, then x is:
(A) $\frac{10m}{m - 10}$ (B) $\frac{5m}{m - 10}$ (C) $\frac{m}{m - 10}$ (D) $\frac{5m}{m - 20}$ (E) not determined by the given information
10. (2008 AMC 10A, Problem #15) Yesterday Han drove 1 hour longer than Ian at an average speed 5 miles per hour faster than Ian. Jan drove 2 hours longer than Ian at an average speed 10 miles per hour faster than Ian. Han drove 70 miles more than Ian. How many more miles did Jan drive than Ian?
(A) 120 (B) 130 (C) 140 (D) 150 (E) 160

Advanced.

11. (2011 AMC 12A, Problem #12) A power boat and a raft both left dock A on a river and headed downstream. The raft drifted at the speed of the river current. The power boat maintained a constant speed with respect to the river. The power boat reached dock B downriver, then immediately turned and traveled back upriver. It eventually met the raft on the river 9 hours after leaving dock A . How many hours did it take the power boat to go from A to B ?
(A) 3 (B) 3.5 (C) 4 (D) 4.5 (E) 5
12. (1954 AHSME, Problem #30) A and B together can do a job in 2 days; B and C can do it in four days; and A and C in $2\frac{2}{5}$ days. The number of days required for A to do the job alone is:
(A) 1 (B) 3 (C) 6 (D) 12 (E) 2.8

13. (2022 AMC 10A, Problem #12) On Halloween 31 children walked into the principal's office asking for candy. They can be classified into three types: Some always lie; some always tell the truth; and some alternately lie and tell the truth. The alternaters arbitrarily choose their first response, either a lie or the truth, but each subsequent statement has the opposite truth value from its predecessor. The principal asked everyone the same three questions in this order.
- “Are you a truth-teller?” The principal gave a piece of candy to each of the 22 children who answered yes.
- “Are you an alternater?” The principal gave a piece of candy to each of the 15 children who answered yes.
- “Are you a liar?” The principal gave a piece of candy to each of the 9 children who answered yes.
- How many pieces of candy in all did the principal give to the children who always tell the truth?
- (A) 7 (B) 12 (C) 21 (D) 27 (E) 31
14. (2000 AMC 10, Problem #22) One morning each member of Angela's family drank an 8-ounce mixture of coffee with milk. The amounts of coffee and milk varied from cup to cup, but were never zero. Angela drank a quarter of the total amount of milk and a sixth of the total amount of coffee. How many people are in the family?
- (A) 3 (B) 4 (C) 5 (D) 6 (E) 7