

SYSTEMS, RATES, AND WORK PROBLEMS FOR AMC 10: SOLUTIONS

ANDY JO

CONTENTS

1. Solutions	2
2. Answer Key	8

1. SOLUTIONS

Beginner.

1. (2012 AMC 10A, Problem #1) Cagney can frost a cupcake every 20 seconds and Lacey can frost a cupcake every 30 seconds. Working together, how many cupcakes can they frost in 5 minutes?

(A) 10 (B) 15 (C) 20 (D) 25 (E) 30

Strategy. Turn each time per cupcake into a rate in cupcakes per minute, then add the rates.

Solution. Cagney frosts a cupcake every 20 seconds, which is 3 cupcakes per minute. Lacey frosts one every 30 seconds, which is 2 cupcakes per minute. Working together their rates add, so they frost $3 + 2 = 5$ cupcakes per minute, and in 5 minutes they frost $5 \cdot 5 = 25$ cupcakes.

Answer: (D) 25. □

2. (2018 AMC 10B, Problem #2) Sam drove 96 miles in 90 minutes. His average speed during the first 30 minutes was 60 mph (miles per hour), and his average speed during the second 30 minutes was 65 mph. What was his average speed, in mph, during the last 30 minutes?

(A) 64 (B) 65 (C) 66 (D) 67 (E) 68

Strategy. Use $d = rt$ for the first two half-hours. The rest of the 96 miles was driven in the last half-hour.

Solution. Each interval is 30 minutes, or $\frac{1}{2}$ hour. In the first interval Sam drove $60 \cdot \frac{1}{2} = 30$ miles, and in the second he drove $65 \cdot \frac{1}{2} = 32.5$ miles. The remaining $96 - 30 - 32.5 = 33.5$ miles took the last $\frac{1}{2}$ hour, so his average speed then was $33.5 \div \frac{1}{2} = 67$ mph.

Answer: (D) 67. □

3. (2006 AMC 10B, Problem #9) Francesca uses 100 grams of lemon juice, 100 grams of sugar, and 400 grams of water to make lemonade. There are 25 calories in 100 grams of lemon juice and 386 calories in 100 grams of sugar. Water contains no calories. How many calories are in 200 grams of her lemonade?

(A) 129 (B) 137 (C) 174 (D) 223 (E) 411

Strategy. The lemonade is mixed evenly, so a 200-gram portion contains the same fraction of the calories as of the weight.

Solution. The lemonade weighs $100 + 100 + 400 = 600$ grams and contains $25 + 386 = 411$ calories, since water has none. A 200-gram portion is $\frac{200}{600} = \frac{1}{3}$ of the lemonade, so it contains $\frac{411}{3} = 137$ calories.

Answer: (B) 137. □

4. (1990 AHSME, Problem #15) Four whole numbers, when added three at a time, give the sums 180, 197, 208, and 222. What is the largest of the four numbers?

(A) 77 (B) 83 (C) 89 (D) 95 (E) cannot be determined from the given information

Strategy. Each number appears in exactly three of the four sums, so add all four sums.

Solution. Let the numbers be a , b , c , and d . Each of them is left out of exactly one of the four sums, so it appears in the other three. Adding the four sums gives

$$3(a + b + c + d) = 180 + 197 + 208 + 222 = 807,$$

so $a + b + c + d = 269$. Each sum of three is the total minus the number left out, so the four numbers are

$$269 - 180 = 89, \quad 269 - 197 = 72, \quad 269 - 208 = 61, \quad 269 - 222 = 47.$$

The largest is 89, the number left out of the smallest sum.

Answer: (C) 89. □

5. (2008 AMC 10A, Problem #11) While Steve and LeRoy are fishing 1 mile from shore, their boat springs a leak, and water comes in at a constant rate of 10 gallons per minute. The boat will sink if it takes in more than 30 gallons of water. Steve starts rowing toward the shore at a constant rate of 4 miles per hour while LeRoy bails water out of the boat. What is the slowest rate, in gallons per minute, at which LeRoy can bail if they are to reach the shore without sinking?

(A) 2 (B) 4 (C) 6 (D) 8 (E) 10

Strategy. Find how long the trip to shore takes. Water comes in at 10 gallons per minute and LeRoy bails it out at r gallons per minute, so the water in the boat grows at the net rate $10 - r$.

Solution. Rowing 1 mile at 4 miles per hour takes $\frac{1}{4}$ hour, which is 15 minutes. If LeRoy bails r gallons per minute, the water in the boat increases by $10 - r$ gallons each minute, so after 15 minutes the boat holds $15(10 - r)$ gallons. The boat does not sink if

$$15(10 - r) \leq 30, \quad \text{that is,} \quad 10 - r \leq 2, \quad \text{or} \quad r \geq 8.$$

The slowest rate that works is 8 gallons per minute.

Answer: (D) 8. □

Intermediate.

6. (2012 AMC 10B, Problem #13) It takes Clea 60 seconds to walk down an escalator when it is not operating, and only 24 seconds to walk down the escalator when it is operating. How many seconds does it take Clea to ride down the operating escalator when she just stands on it?

(A) 36 (B) 40 (C) 42 (D) 48 (E) 52

Strategy. Let L be the length of the escalator. When Clea walks on the moving escalator, her walking speed and the speed of the escalator add.

Solution. On the stopped escalator Clea covers L in 60 seconds, so her walking speed is $\frac{L}{60}$ per second. On the operating escalator her speed is her walking speed plus the speed e of the escalator, and she covers L in 24 seconds:

$$\frac{L}{60} + e = \frac{L}{24}, \quad \text{so} \quad e = \frac{L}{24} - \frac{L}{60} = \frac{5L - 2L}{120} = \frac{L}{40}.$$

Standing still, she moves at the speed of the escalator, $\frac{L}{40}$ per second, so the ride takes 40 seconds.

Answer: (B) 40. □

7. (2002 AMC 10A, Problem #16) If $a + 1 = b + 2 = c + 3 = d + 4 = a + b + c + d + 5$, then $a + b + c + d$ is

(A) -5 (B) $-\frac{10}{3}$ (C) $-\frac{7}{3}$ (D) $\frac{5}{3}$ (E) 5

Strategy. The five expressions are equal, so call their common value k . Write a , b , c , and d in terms of k and add them.

Solution. Let k be the common value and $S = a + b + c + d$. Then $a = k - 1$, $b = k - 2$, $c = k - 3$, and $d = k - 4$, so adding gives

$$S = 4k - 10.$$

The last expression gives $k = S + 5$. Substituting,

$$S = 4(S + 5) - 10 = 4S + 10, \quad \text{so} \quad 3S = -10 \quad \text{and} \quad S = -\frac{10}{3}.$$

Check: $k = \frac{5}{3}$, so $a = \frac{2}{3}$, $b = -\frac{1}{3}$, $c = -\frac{4}{3}$, and $d = -\frac{7}{3}$, whose sum is $-\frac{10}{3}$, and $-\frac{10}{3} + 5 = \frac{5}{3}$.

Answer: (B) $-\frac{10}{3}$. □

8. (1986 AHSME, Problem #12) John scores 93 on this year's AHSME. Had the old scoring system still been in effect, he would score only 84 for the same answers. How many questions does he leave unanswered? (In the new scoring system that year, one received 5 points for each correct answer, 0 points for each wrong answer, and 2 points for each problem left unanswered. In the previous scoring system, one started with 30 points, received 4 more for each correct answer, lost 1 point for each wrong answer, and neither gained nor lost points for unanswered questions. There are 30 questions in the 1986 AHSME.)
 (A) 6 (B) 9 (C) 11 (D) 14 (E) not uniquely determined

Strategy. Use three unknowns: the numbers of correct, wrong, and unanswered questions. The number of questions and the two scores give three linear equations, and the question asks only for the number unanswered.

Solution. Let c , w , and u be the numbers of correct, wrong, and unanswered questions. Then

$$c + w + u = 30,$$

$$5c + 2u = 93,$$

$$30 + 4c - w = 84.$$

The third equation says $4c - w = 54$. Adding it to the first equation eliminates w :

$$5c + u = 84.$$

Subtracting this from the second equation eliminates c and gives $u = 93 - 84 = 9$. Then $5c = 75$, so $c = 15$ and $w = 6$. Check: $15 + 6 + 9 = 30$, $5 \cdot 15 + 2 \cdot 9 = 93$, and $30 + 4 \cdot 15 - 6 = 84$.

Answer: (B) 9. □

9. (1967 AHSME, Problem #8) To m ounces of an $m\%$ solution of acid, x ounces of water are added to yield an $(m - 10)\%$ solution. If $m > 25$, then x is:
 (A) $\frac{10m}{m - 10}$ (B) $\frac{5m}{m - 10}$ (C) $\frac{m}{m - 10}$ (D) $\frac{5m}{m - 20}$ (E) not determined
 by the given information

Strategy. Adding water does not change the amount of acid. Write the amount of acid before and after, and set them equal.

Solution. The original m ounces are $m\%$ acid, so they contain $\frac{m}{100} \cdot m = \frac{m^2}{100}$ ounces of acid. After x ounces of water are added, there are $m + x$ ounces of solution that is $(m - 10)\%$ acid, so the acid is $\frac{m-10}{100}(m + x)$ ounces. The acid is unchanged, so

$$m^2 = (m - 10)(m + x) = m^2 + mx - 10m - 10x.$$

This gives $mx - 10x = 10m$, that is, $x(m - 10) = 10m$. Since $m > 25$, the factor $m - 10$ is not zero, and

$$x = \frac{10m}{m - 10}.$$

Check with $m = 50$: the 50 ounces contain 25 ounces of acid, $x = \frac{500}{40} = 12.5$, and $\frac{25}{62.5} = 40\% = (50 - 10)\%$.

Answer: (A) $\frac{10m}{m - 10}$. □

10. (2008 AMC 10A, Problem #15) Yesterday Han drove 1 hour longer than Ian at an average speed 5 miles per hour faster than Ian. Jan drove 2 hours longer than Ian at an average speed 10 miles per hour faster than Ian. Han drove 70 miles more than Ian. How many more miles did Jan drive than Ian?

(A) 120 (B) 130 (C) 140 (D) 150 (E) 160

Strategy. Let Ian drive for t hours at s miles per hour. Han's information does not determine s and t , but it determines $s + 5t$. Expand Jan's distance and look for the same combination.

Solution. Ian drove st miles. Han drove $(s + 5)(t + 1) = st + s + 5t + 5$ miles, which is 70 more than Ian, so

$$s + 5t + 5 = 70, \quad \text{so} \quad s + 5t = 65.$$

Jan drove $(s + 10)(t + 2) = st + 2s + 10t + 20$ miles, which is

$$2s + 10t + 20 = 2(s + 5t) + 20 = 2 \cdot 65 + 20 = 150$$

miles more than Ian.

Answer: (D) 150. □

Advanced.

11. (2011 AMC 12A, Problem #12) A power boat and a raft both left dock A on a river and headed downstream. The raft drifted at the speed of the river current. The power boat maintained a constant speed with respect to the river. The power boat reached dock B downriver, then immediately turned and traveled back upriver. It eventually met the raft on the river 9 hours after leaving dock A . How many hours did it take the power boat to go from A to B ?

(A) 3 (B) 3.5 (C) 4 (D) 4.5 (E) 5

Strategy. Let c be the speed of the current and b the speed of the boat in still water, so the boat moves downstream at $b + c$ and upstream at $b - c$. Write the positions of the raft and the boat at the meeting time and set them equal.

Solution. Let the boat take t hours from A to B , and measure positions in miles downstream from dock A . After 9 hours the raft, drifting at speed c , is at $9c$. The boat went downstream for t hours, reaching $(b + c)t$, and then upstream for $9 - t$ hours, so it is at $(b + c)t - (b - c)(9 - t)$. They meet, so

$$(b + c)t - (b - c)(9 - t) = 9c.$$

The left side expands to $bt + ct - 9b + bt + 9c - ct = 2bt - 9b + 9c$, so $2bt - 9b = 0$, that is, $b(2t - 9) = 0$. The boat could travel upriver, so $b > c \geq 0$, and therefore $t = \frac{9}{2} = 4.5$.

The current does not affect the answer. Relative to the water, the raft does not move and the boat moves at the same speed b in both directions, so the boat takes as long to move away from the raft as to come back to it.

Answer: (D) 4.5. □

12. (1954 AHSME, Problem #30) A and B together can do a job in 2 days; B and C can do it in four days; and A and C in $2\frac{2}{5}$ days. The number of days required for A to do the job alone is:

(A) 1 (B) 3 (C) 6 (D) 12 (E) 2.8

Strategy. Let a , b , and c be the rates in jobs per day. Each pair gives one equation, and each rate appears in two of the three equations, so add all three.

Solution. A pair that finishes the job in t days has rate $\frac{1}{t}$. Since $2\frac{2}{5} = \frac{12}{5}$,

$$a + b = \frac{1}{2}, \quad b + c = \frac{1}{4}, \quad a + c = \frac{5}{12}.$$

Adding the three equations gives $2(a + b + c) = \frac{6}{12} + \frac{3}{12} + \frac{5}{12} = \frac{7}{6}$, so $a + b + c = \frac{7}{12}$. Subtracting $b + c = \frac{1}{4}$ leaves

$$a = \frac{7}{12} - \frac{3}{12} = \frac{1}{3}.$$

So A alone does $\frac{1}{3}$ of the job per day and needs 3 days. As a check, $b = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$ and $c = \frac{1}{4} - \frac{1}{6} = \frac{1}{12}$, and $a + c = \frac{1}{3} + \frac{1}{12} = \frac{5}{12}$.

Answer: (B) 3. □

13. (2022 AMC 10A, Problem #12) On Halloween 31 children walked into the principal's office asking for candy. They can be classified into three types: Some always lie; some always tell the truth; and some alternately lie and tell the truth. The alternaters arbitrarily choose their first response, either a lie or the truth, but each subsequent statement has the opposite truth value from its predecessor. The principal asked everyone the same three questions in this order.

"Are you a truth-teller?" The principal gave a piece of candy to each of the 22 children who answered yes.

"Are you an alternater?" The principal gave a piece of candy to each of the 15 children who answered yes.

"Are you a liar?" The principal gave a piece of candy to each of the 9 children who answered yes.

How many pieces of candy in all did the principal give to the children who always tell the truth?

(A) 7 (B) 12 (C) 21 (D) 27 (E) 31

Strategy. Split the alternaters into those who lie first and those who tell the truth first. Work out how each type answers each question. Then each count of yes answers is a linear equation.

Solution. Let T and L be the numbers of truth-tellers and liars. Let x be the number of alternaters who lie, tell the truth, and lie, and y the number who tell the truth, lie, and

tell the truth. Their answers are:

Type	Truth-teller?	Alternater?	Liar?
Truth-teller	yes	no	no
Liar	yes	yes	no
Alternater who lies first	yes	yes	yes
Alternater who tells the truth first	no	no	no

For example, a liar is not an alternater, so the true answer to the second question is no, and the liar says yes. An alternater who lies first tells the truth on the second question and says yes, then lies on the third question, where the true answer is no, and says yes. Counting the yes answers to the three questions,

$$T + L + x = 22, \quad L + x = 15, \quad x = 9.$$

So $L = 6$ and $T = 22 - 15 = 7$. A truth-teller answers yes only to the first question, so the truth-tellers received 7 pieces of candy in all. (The other $31 - 7 - 6 - 9 = 9$ children are alternaters who tell the truth first.)

Answer: (A) 7. □

14. (2000 AMC 10, Problem #22) One morning each member of Angela's family drank an 8-ounce mixture of coffee with milk. The amounts of coffee and milk varied from cup to cup, but were never zero. Angela drank a quarter of the total amount of milk and a sixth of the total amount of coffee. How many people are in the family?
(A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

Strategy. Let M and C be the total amounts of milk and coffee. Angela's share of all the liquid is a weighted average of her share of the milk, $\frac{1}{4}$, and her share of the coffee, $\frac{1}{6}$.

Solution. Let n be the number of people. The family drank $8n$ ounces in all, so $M + C = 8n$, and Angela drank

$$\frac{1}{4}M + \frac{1}{6}C = 8$$

ounces. Her share of all the liquid is $\frac{8}{8n} = \frac{1}{n}$, so

$$\frac{1}{n} = \frac{\frac{1}{4}M + \frac{1}{6}C}{M + C},$$

which is a weighted average of $\frac{1}{4}$ and $\frac{1}{6}$ with weights M and C . Every cup had some milk and some coffee, so $M > 0$ and $C > 0$. By the weighted average rule (Section 4.2 of the handout), $\frac{1}{6} < \frac{1}{n} < \frac{1}{4}$. Hence $4 < n < 6$, so $n = 5$.

A family of 5 is possible. Then $M + C = 40$ and $3M + 2C = 96$ (the equation for Angela's cup times 12), so $M = 16$ and $C = 24$. Angela can drink 4 ounces of milk and 4 ounces of coffee, and each of the other four people can drink 3 ounces of milk and 5 ounces of coffee.

Answer: (C) 5. □

2. ANSWER KEY

Problem	Letter	Value
1	D	25
2	D	67
3	B	137
4	C	89
5	D	8
6	B	40
7	B	$-10/3$
8	B	9
9	A	$10m/(m - 10)$
10	D	150
11	D	4.5
12	B	3
13	A	7
14	C	5