# SYSTEMS, RATES, AND WORK PROBLEMS FOR AMC 10: PRACTICE SET SOLUTIONS

ANDY JO

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## 1. Solutions

**Beginner.**

**1.** (2021 AMC 10B, Problem #3) In an after-school program for juniors and seniors, there is a debate team with an equal number of students from each class on the team. Among the 28 students in the program, 25% of the juniors and 10% of the seniors are on the debate team. How many juniors are in the program?

**(A)** 5 **(B)** 6 **(C)** 8 **(D)** 11 **(E)** 20

*Strategy.* Let $J$ and $S$ be the numbers of juniors and seniors. Write one equation for the total of 28 students and one for the equal numbers of juniors and seniors on the debate team.

*Solution.* The program has $J + S = 28$ students, and the debate team has equally many juniors and seniors, so $\frac{J}{4} = \frac{S}{10}$, that is, $S = \frac{5J}{2}$. Substituting,

$$J + \frac{5J}{2} = \frac{7J}{2} = 28,$$

so $J = 8$. As a check, $S = 20$, and the team has 2 juniors and 2 seniors.

**Answer:** (C) 8. □

**2.** (2022 AMC 10A, Problem #3) The sum of three numbers is 96. The first number is 6 times the third number, and the third number is 40 less than the second number. What is the absolute value of the difference between the first and second numbers?

**(A)** 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

*Strategy.* Name the third number and write the other two in terms of it. Then the sum gives one equation in one unknown.

*Solution.* Let the third number be $c$. The first number is $6c$ and the second is $c+40$. Their sum is

$$6c + (c + 40) + c = 8c + 40 = 96,$$

so $c = 7$. The first number is 42 and the second is 47, and $|42 - 47| = 5$.

**Answer:** (E) 5. □

**3.** (2021 Fall AMC 10A, Problem #4) Mr. Lopez has a choice of two routes to get to work. Route A is 6 miles long, and his average speed along this route is 30 miles per hour. Route B is 5 miles long, and his average speed along this route is 40 miles per hour, except for a $\frac{1}{2}$-mile stretch in a school zone where his average speed is 20 miles per hour. By how many minutes is Route B quicker than Route A?

**(A)** $2\frac{3}{4}$ **(B)** $3\frac{3}{4}$ **(C)** $4\frac{1}{2}$ **(D)** $5\frac{1}{2}$ **(E)** $6\frac{3}{4}$

*Strategy.* Time equals distance divided by speed. Route B has two parts with different speeds, so time each part separately and add.

*Solution.* Route A takes $\frac{6}{30} = \frac{1}{5}$ hour, which is 12 minutes. On Route B, the school zone takes $\frac{1/2}{20} = \frac{1}{40}$ hour, which is 1.5 minutes, and the other 4.5 miles take $\frac{4.5}{40}$ hour, which is $\frac{4.5}{40} \cdot 60 = 6.75$ minutes. So Route B takes $1.5 + 6.75 = 8.25$ minutes, and it is quicker by $12 - 8.25 = 3.75 = 3\frac{3}{4}$ minutes.

**Answer:** (B) $3\frac{3}{4}$. □

**4.** (2019 AMC 10B, Problem #3) In a high school with 500 students, 40% of the seniors play a musical instrument, while 30% of the non-seniors do not play a musical instrument. In all, 46.8% of the students do not play a musical instrument. How many non-seniors play a musical instrument?

**(A)** 66 **(B)** 154 **(C)** 186 **(D)** 220 **(E)** 266

*Strategy.* Let $s$ be the number of seniors. Write the number of students who do not play an instrument in terms of $s$ and set it equal to 46.8% of 500.

*Solution.* Let $s$ be the number of seniors, so there are $500 - s$ non-seniors. Since 40% of the seniors play, 60% of them do not, and 30% of the non-seniors do not play. In all, $0.468 \cdot 500 = 234$ students do not play, so

$$0.6s + 0.3(500 - s) = 234 \iff 0.3s + 150 = 234 \iff s = 280.$$

There are $500-280 = 220$ non-seniors, and 70% of them play an instrument: $0.7 \cdot 220 = 154$.

The overall rate of not playing, 46.8%, is the weighted average of the rates 60% for seniors and 30% for non-seniors, weighted by the numbers of seniors and non-seniors. So the balance $s(60 - 46.8) = (500 - s)(46.8 - 30)$ also gives the group sizes directly: $s : (500 - s) = 16.8 : 13.2 = 14 : 11$, so $s = \frac{14}{25} \cdot 500 = 280$ again.

**Answer:** (B) 154. □

**5.** (2017 AMC 10A, Problem #4) Mia is "helping" her mom pick up 30 toys that are strewn on the floor. Mia's mom manages to put 3 toys into the toy box every 30 seconds, but each time immediately after those 30 seconds have elapsed, Mia takes 2 toys out of the box. How much time, in minutes, will it take Mia and her mom to put all 30 toys into the box for the first time?

**(A)** 13.5 **(B)** 14 **(C)** 14.5 **(D)** 15 **(E)** 15.5

*Strategy.* In each half-minute the mother adds 3 toys and then Mia removes 2, so the box gains 1 toy per half-minute. The box first holds all 30 toys at the end of a half-minute, right after the mother's 3 toys go in and before Mia removes any.

*Solution.* Measure time in half-minutes. During each half-minute the mother puts 3 toys into the box, and right after it Mia takes 2 out, so each complete half-minute adds $3-2 = 1$ toy to the box. After $k$ complete half-minutes the box holds $k$ toys, and at the end of the next half-minute, before Mia takes any out, it holds $k + 3$ toys.

The count $k + 3$ first equals 30 when $k = 27$, that is, at the end of half-minute number 28; at the end of each earlier half-minute the box holds at most 29 toys. So all 30 toys are in the box for the first time after 28 half-minutes, which is $28 \cdot 30 = 840$ seconds, or 14 minutes.

Dividing 30 toys by the net rate of 1 toy per half-minute would give 15 minutes instead. That count misses that the last 3 toys go in before Mia can take any of them out.

**Answer:** (B) 14. □

**Intermediate.**

**6.** (1959 AHSME, Problem #30) $A$ can run around a circular track in 40 seconds. $B$, running in the opposite direction, meets $A$ every 15 seconds. What is $B$'s time to run around the track, expressed in seconds?

**(A)** $12\frac{1}{2}$ **(B)** 24 **(C)** 25 **(D)** $27\frac{1}{2}$ **(E)** 55

*Strategy.* Runners going in opposite directions approach each other at the sum of their speeds. Measure distance in laps: between two meetings, the two runners together cover exactly one lap.

*Solution.* After a meeting, $A$ and $B$ run away from each other in opposite directions around the track, and they meet again when the distances they have run add up to one full lap. This takes 15 seconds, so their combined speed is $\frac{1}{15}$ lap per second. The speed of $A$ is $\frac{1}{40}$ lap per second, so the speed of $B$ is

$$\frac{1}{15} - \frac{1}{40} = \frac{8-3}{120} = \frac{1}{24}$$

lap per second. Hence $B$ runs around the track in 24 seconds.

**Answer:** (B) 24. □

**7.** (2006 AMC 10B, Problem #13) Joe and JoAnn each bought 12 ounces of coffee in a 16-ounce cup. Joe drank 2 ounces of his coffee and then added 2 ounces of cream. JoAnn added 2 ounces of cream, stirred the coffee well, and then drank 2 ounces. What is the resulting ratio of the amount of cream in Joe's coffee to that in JoAnn's coffee?

**(A)** $\frac{6}{7}$ **(B)** $\frac{13}{14}$ **(C)** 1 **(D)** $\frac{14}{13}$ **(E)** $\frac{7}{6}$

*Strategy.* Track the cream in each cup. Joe drinks only coffee, so he keeps all of his cream. JoAnn's cup is well stirred, so the part she drinks contains cream in the same proportion as her whole cup.

*Solution.* Joe drinks 2 ounces of coffee before he adds any cream, so his cup ends with all 2 ounces of cream.

JoAnn adds 2 ounces of cream to 12 ounces of coffee, so her cup holds 14 ounces, and the concentration of cream is $\frac{2}{14} = \frac{1}{7}$. The mixture is well stirred, so the 2 ounces she drinks are also $\frac{1}{7}$ cream: they contain $2 \cdot \frac{1}{7} = \frac{2}{7}$ ounce of cream. Her cup keeps

$$2 - \frac{2}{7} = \frac{12}{7}$$

ounces of cream. The ratio of Joe's cream to JoAnn's is

$$\frac{2}{12/7} = \frac{14}{12} = \frac{7}{6}.$$

**Answer:** (E) $\frac{7}{6}$. □

**8.** (1958 AHSME, Problem #24) A man travels $m$ feet due north at 2 minutes per mile. He returns due south to his starting point at 2 miles per minute. The average rate in miles per hour for the entire trip is:

**(A)** 75 **(B)** 48 **(C)** 45 **(D)** 24
**(E)** impossible to determine without knowing the value of $m$

*Strategy.* Convert both rates to miles per hour. The two legs have the same length, so the average speed is $\frac{2uv}{u+v}$, where $u$ and $v$ are the speeds on the two legs, and the length of the legs cancels.

*Solution.* A pace of 2 minutes per mile is $\frac{1}{2}$ mile per minute, which is 30 miles per hour. A rate of 2 miles per minute is 120 miles per hour. Let $d = \frac{m}{5280}$ be the length of each leg in

miles. The trip north takes $\frac{d}{30}$ hours and the trip south takes $\frac{d}{120}$ hours. The average rate is the total distance divided by the total time:

$$\frac{2d}{\frac{d}{30}+\frac{d}{120}} = \frac{2d}{\frac{5d}{120}} = \frac{240}{5} = 48$$

miles per hour, which is the value of $\frac{2uv}{u+v}$, the average speed over two equal distances, at $u = 30$ and $v = 120$. The length $d$ cancels, so the average rate is 48 miles per hour for every value of $m$.

**Answer:** (B) 48. □

**9.** (1956 AHSME, Problem #47) An engineer said he could finish a highway section in 3 days with his present supply of a certain type of machine. However, with 3 more of these machines the job could be done in 2 days. If the machines all work at the same rate, how many days would it take to do the job with one machine?

**(A)** 6 **(B)** 12 **(C)** 15 **(D)** 18 **(E)** 36

*Strategy.* Rates add, so $n$ identical machines work $n$ times as fast as one. Write the whole job as rate times time for both groups of machines.

*Solution.* Let $n$ be the number of machines the engineer has, and let one machine do $r$ of the job per day. The rates add, so $n$ machines do $nr$ of the job per day and $n+3$ machines do $(n+3)r$. Each group does the whole job, so

$$3nr = 1 \qquad \text{and} \qquad 2(n+3)r = 1.$$

Setting the two left sides equal and dividing by $r$ gives $3n = 2n + 6$, so $n = 6$. Then $18r = 1$, so $r = \frac{1}{18}$, and one machine needs 18 days. Check: 9 machines do $\frac{9}{18} = \frac{1}{2}$ of the job per day, so they finish in 2 days.

**Answer:** (D) 18. □

**10.** (1973 AHSME, Problem #24) The check for a luncheon of 3 sandwiches, 7 cups of coffee and one piece of pie came to \$3.15. The check for a luncheon consisting of 4 sandwiches, 10 cups of coffee and one piece of pie came to \$4.20 at the same place. The cost of a luncheon consisting of one sandwich, one cup of coffee and one piece of pie at the same place will come to

**(A)** \$1.70 **(B)** \$1.65 **(C)** \$1.20 **(D)** \$1.05 **(E)** \$0.95

*Strategy.* Two equations cannot determine three prices, but the question asks only for their sum. Find multiples of the two equations that add up to that sum.

*Solution.* Let $s$, $c$, and $p$ be the prices in dollars of a sandwich, a cup of coffee, and a piece of pie. The two checks give

$$3s + 7c + p = 3.15, \qquad 4s + 10c + p = 4.20.$$

Look for numbers $a$ and $b$ with $a(3s + 7c + p) + b(4s + 10c + p) = s + c + p$. Matching the coefficients of $s$, $c$, and $p$ gives

$$3a + 4b = 1, \qquad 7a + 10b = 1, \qquad a + b = 1.$$

The last equation gives $b = 1 - a$, and then the first becomes $3a + 4 - 4a = 1$, so $a = 3$ and $b = -2$. These values also satisfy the middle equation, since $21 - 20 = 1$. Therefore

$$s + c + p = 3(3.15) - 2(4.20) = 9.45 - 8.40 = 1.05.$$

The separate prices are not determined. For example, both

$$(s, c, p) = (0.45, 0.20, 0.40) \qquad \text{and} \qquad (s, c, p) = (0.15, 0.30, 0.60)$$

fit the two checks, and in both cases $s + c + p = 1.05$.

**Answer:** (D) \$1.05. □

**Advanced.**

**11.** (1986 AIME, Problem #4) Determine $3x_4 + 2x_5$ if $x_1$, $x_2$, $x_3$, $x_4$, and $x_5$ satisfy the system

$$\begin{aligned} 2x_1 + x_2 + x_3 + x_4 + x_5 &= 6, \\ x_1 + 2x_2 + x_3 + x_4 + x_5 &= 12, \\ x_1 + x_2 + 2x_3 + x_4 + x_5 &= 24, \\ x_1 + x_2 + x_3 + 2x_4 + x_5 &= 48, \\ x_1 + x_2 + x_3 + x_4 + 2x_5 &= 96. \end{aligned}$$

*Strategy.* Each left side is the sum of all five unknowns plus one more copy of one of them. Every unknown appears equally often, so add all five equations to find the total.

*Solution.* Let $S = x_1 + x_2 + x_3 + x_4 + x_5$. The five equations say

$$S + x_1 = 6, \qquad S + x_2 = 12, \qquad S + x_3 = 24, \qquad S + x_4 = 48, \qquad S + x_5 = 96.$$

Adding them gives $5S + (x_1 + x_2 + x_3 + x_4 + x_5) = 6S$ on the left, so

$$6S = 6 + 12 + 24 + 48 + 96 = 186$$

and $S = 31$. Subtracting $S$ from the fourth and fifth equations gives $x_4 = 48 - 31 = 17$ and $x_5 = 96 - 31 = 65$. Therefore

$$3x_4 + 2x_5 = 3 \cdot 17 + 2 \cdot 65 = 51 + 130 = 181.$$

**Answer:** 181. □

**12.** (1973 AHSME, Problem #10) If $n$ is a real number, then the simultaneous system

$$\begin{cases} nx + y = 1, \\ ny + z = 1, \\ x + nz = 1 \end{cases}$$

has no solution if and only if $n$ is equal to

**(A)** $-1$ **(B)** $0$ **(C)** $1$ **(D)** $0$ or $1$ **(E)** $\frac{1}{2}$

*Strategy.* Eliminate $y$ and $z$ by substitution to get one equation in $x$. If that equation is a contradiction, such as $0 = 3$, there is no solution; otherwise it gives $x$, and then $y$ and $z$.

*Solution.* The first equation gives $y = 1 - nx$. Substituting this into the second equation gives

$$z = 1 - ny = 1 - n + n^2 x.$$

Substituting this into the third equation eliminates $y$ and $z$:

$$x + n(1 - n + n^2 x) = 1, \qquad \text{that is,} \qquad (1 + n^3)x = 1 - n + n^2.$$

If $n = -1$, this equation reads $0 \cdot x = 3$, a contradiction, so the system has no solution.

If $n \neq -1$, then $n^3 \neq -1$, because $-1$ is the only real number whose cube is $-1$. So $1 + n^3 \neq 0$, and the last equation gives exactly one value of $x$. The first two equations then give $y$ and $z$, and the third equation holds because, with these $y$ and $z$, it is equivalent to

the equation that gave $x$. So the system has a solution for every real $n$ other than $-1$. For example, $n = 0$ gives $x = y = z = 1$, and $n = 1$ gives $x = y = z = \frac{1}{2}$.

Adding all three equations shows the contradiction at once. Each unknown appears once with coefficient $n$ and once with coefficient 1, so $(n+1)(x+y+z) = 3$, which reads $0 = 3$ when $n = -1$.

**Answer:** (A) $-1$. □

**13.** (1955 AHSME, Problem #50) In order to pass $B$ going 40 mph on a two-lane highway $A$, going 50 mph, must gain 30 feet. Meantime, $C$, 210 feet from $A$, is headed toward him at 50 mph. If $B$ and $C$ maintain their speeds, then, in order to pass safely, $A$ must increase his speed by:

**(A)** 30 mph **(B)** 10 mph **(C)** 5 mph **(D)** 15 mph **(E)** 3 mph

*Strategy.* Follow two gaps. Car $A$ gains on $B$ at the difference of their speeds, while $A$ and $C$ close the distance between them at the sum of their speeds. The pass is safe if $A$ gains the 30 feet before that distance closes.

*Solution.* Let $v$ be the speed of $A$ in miles per hour after the increase, with $v > 40$. In miles, the two distances are $\frac{30}{5280}$ and $\frac{210}{5280}$.

Relative to $B$, car $A$ moves at $v - 40$ miles per hour, so it gains the 30 feet in $\frac{30}{5280(v-40)}$ hours. Cars $A$ and $C$ move toward each other, so the 210 feet between them shrink at $v + 50$ miles per hour and are gone after $\frac{210}{5280(v+50)}$ hours. The pass is safe when the first time is at most the second. The factor $\frac{1}{5280}$ cancels, so the condition is

$$\frac{30}{v-40} \le \frac{210}{v+50}.$$

Both denominators are positive, so multiplying by $(v-40)(v+50)$ and dividing by 30 gives $v + 50 \le 7(v - 40) = 7v - 280$. Hence $6v \ge 330$, that is, $v \ge 55$.

At the present speed of 50 miles per hour the condition fails, since $\frac{30}{10} = 3$ is more than $\frac{210}{100} = 2.1$. The slowest safe speed is 55 miles per hour, so $A$ must increase his speed by 5 miles per hour.

**Answer:** (C) 5 mph. □

**14.** (2011 AIME I, Problem #1) Jar $A$ contains four liters of a solution that is 45% acid. Jar $B$ contains five liters of a solution that is 48% acid. Jar $C$ contains one liter of a solution that is $k$% acid. From jar $C$, $\frac{m}{n}$ liters of the solution is added to jar $A$, and the remainder of the solution in jar $C$ is added to jar $B$. At the end both jar $A$ and jar $B$ contain solutions that are 50% acid. Given that $m$ and $n$ are relatively prime positive integers, find $k + m + n$.

*Strategy.* Track the liters of acid. Jars $A$ and $B$ each give one equation, amount of acid = concentration × total. Adding the two equations removes the amount poured into jar $A$ and gives $k$.

*Solution.* Jar $A$ holds $0.45 \cdot 4 = 1.8$ liters of acid, jar $B$ holds $0.48 \cdot 5 = 2.4$ liters, and jar $C$ holds $\frac{k}{100}$ liter. Write $x$ for $\frac{m}{n}$: jar $A$ receives $x$ liters of the solution from jar $C$, and jar $B$ receives the other $1 - x$ liters. When solutions are combined, the amounts of acid add and the volumes add. At the end each of jars $A$ and $B$ is half acid, so

$$1.8 + \frac{k}{100}\,x = \frac{1}{2}(4 + x), \qquad 2.4 + \frac{k}{100}(1 - x) = \frac{1}{2}(5 + 1 - x).$$

Adding the two equations eliminates $x$:

$$4.2 + \frac{k}{100} = \frac{1}{2} \cdot 10 = 5,$$

so $\frac{k}{100} = 0.8$ and $k = 80$. (The sum says that the ten liters of all three jars together are half acid.) With $k = 80$, the equation for jar $A$ becomes $1.8 + 0.8x = 2 + 0.5x$, so $0.3x = 0.2$ and $x = \frac{2}{3}$. Check with jar $B$: it ends with $2.4 + 0.8 \cdot \frac{1}{3} = \frac{8}{3}$ liters of acid in $6 - \frac{2}{3} = \frac{16}{3}$ liters of solution, which is half. So $m = 2$ and $n = 3$, which are relatively prime, and $k + m + n = 80 + 2 + 3 = 85$.

**Answer:** 85. □

**15.** (2012 AMC 10A, Problem #19) Paula the painter and her two helpers each paint at constant, but different, rates. They always start at 8:00 AM, and all three always take the same amount of time to eat lunch.
On Monday the three of them painted 50% of a house, quitting at 4:00 PM. On Tuesday, when Paula wasn't there, the two helpers painted only 24% of the house and quit at 2:12 PM. On Wednesday Paula worked by herself and finished the house by working until 7:12 P.M.
How long, in minutes, was each day's lunch break?
**(A)** 30 **(B)** 36 **(C)** 42 **(D)** 48 **(E)** 60

*Strategy.* Let $p$ be Paula's rate and $h$ the helpers' combined rate, in houses per hour, and let lunch last $L$ hours. Each day gives one equation, work = rate × time. Tuesday's and Wednesday's work add up to Monday's, and comparing the equations cancels the terms with $L$.

*Solution.* Each day starts at 8:00 AM. Monday ends 8 hours later, Tuesday 6.2 hours later, and Wednesday 11.2 hours later, and the working times are these minus $L$. On Wednesday Paula paints the rest of the house, which is $100\% - 50\% - 24\% = 26\%$ of it. Since work equals rate times time,

$$(8 - L)(p + h) = 0.50, \qquad (6.2 - L)h = 0.24, \qquad (11.2 - L)p = 0.26.$$

Expanded, these read

$$8p + 8h - Lp - Lh = 0.50, \qquad 6.2h - Lh = 0.24, \qquad 11.2p - Lp = 0.26.$$

Treat the products $Lp$ and $Lh$ as unknowns of their own, so that these equations are linear. Since $0.24 + 0.26 = 0.50$, subtracting the first equation from the sum of the other two cancels $Lp$ and $Lh$ and leaves $3.2p - 1.8h = 0$, so $h = \frac{16}{9}p$. Substituting this into Tuesday's equation gives $(6.2 - L) \cdot \frac{16}{9}p = 0.24$, so

$$6.2p - Lp = 0.24 \cdot \frac{9}{16} = 0.135.$$

Subtracting this from Wednesday's equation $11.2p - Lp = 0.26$ gives $5p = 0.125$, so $p = \frac{1}{40}$. Then $11.2 - L = 0.26 \cdot 40 = 10.4$, so $L = 0.8$ hour, which is 48 minutes.

Check: $h = \frac{16}{9} \cdot \frac{1}{40} = \frac{2}{45}$, and on Monday the three paint $7.2\left(\frac{1}{40} + \frac{2}{45}\right) = 7.2 \cdot \frac{25}{360} = 0.5$ of the house.

**Answer:** (D) 48. □

## 2. Answer Key

| Problem | Letter | Value |
|---|---|---|
| 1 | C | 8 |
| 2 | E | 5 |
| 3 | B | $3\frac{3}{4}$ |
| 4 | B | 154 |
| 5 | B | 14 |
| 6 | B | 24 |
| 7 | E | $\frac{7}{6}$ |
| 8 | B | 48 |
| 9 | D | 18 |
| 10 | D | \$1.05 |
| 11 | | 181 |
| 12 | A | $-1$ |
| 13 | C | 5 mph |
| 14 | | 85 |
| 15 | D | 48 |